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THE ANNALS of MATHEMATICAL STATISTICS

(FOUNDED BY H. C. CARVER)

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THE ANNALS OF MATHEMATICAL STATISTICS

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ON SOME USEFUL "INEFFICIENT" STATISTICS

By Frederick Mosteller

Princeton University

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Summary. Several statistical techniques are proposed for economically analyzing large masses of data by means of punched-card equipment; most of these techniques require only a counting sorter. The methods proposed are designed especially for situations where data are inexpensive compared to the cost of analysis by means of statistically "efficient" or "most powerful" procedures. The principal technique is the use of functions of order statistics, which we call *systematic statistics*.

It is demonstrated that certain order statistics are asymptotically jointly distributed according to the normal multivariate law.

For large samples drawn from normally distributed variables we describe and give the efficiencies of rapid methods:

i) for estimating the mean by using 1, 2, $\cdots$, 10 suitably chosen order statistics; (cf. p. 386)

ii) for estimating the standard deviation by using 2, 4, or 8 suitably chosen order statistics; (cf. p. 389)

iii) for estimating the correlation coefficient whether other parameters of the normal bivariate distribution are known or not (three sorting and three counting operations are involved) (cf. p. 394).

The efficiencies of procedures ii) and iii) are compared with the efficiencies of other estimates which do not involve sums of squares or products.

1. Introduction. The purpose of this paper is to contribute some results concerning the use of order statistics in the statistical analysis of large masses of data. The present results deal particularly with estimation when normally distributed variables are present. Solutions to all problems considered have

been especially designed for use with punched-card equipment although for most of the results a counting sorter is adequate.

Until recently mathematical statisticians have spent a great deal of effort developing "efficient statistics" and "most powerful tests." This concentration of effort has often led to neglect of questions of economy. Indeed some may have confused the meaning of technical statistical terms "efficient" and "efficiency" with the layman's concept of their meaning. No matter how much energetic activity is put into analysis and computation, it seems reasonable to inquire whether the output of information is comparable in value to the input measured in dollars, man-hours, or otherwise. Alternatively we may inquire whether comparable results could have been obtained by smaller expenditures. In some fields where statistics is widely used, the collection of large masses of data is inexpensive compared to the cost of analysis. Often the value of the statistical information gleaned from the sample decreases rapidly as the time between collection of data and action on their interpretation increases. Under these conditions, it is important to have quick, inexpensive methods for analyzing data, because economy demands militate against the use of lengthy, costly (even if more precise) statistical methods. A good example of a practical alternative is given by the control chart method in the field of industrial quality control. The sample range rather than the sample standard deviation is used almost invariably in spite of its larger variance. One reason is that, after brief training, persons with slight arithmetical knowledge can compute the range quickly and accurately, while the more complicated formula for the sample standard deviation would create a permanent stumbling block. Largely as a result of simplifying and routinizing statistical methods, industry now handles large masses of data on production adequately and profitably. Although the sample standard deviation can give a statistically more efficient estimate of the population standard deviation, if collection of data is inexpensive compared to cost of analysis and users can compute a dozen ranges to one standard deviation, it is easy to see that economy lies with the less efficient statistic.

It should not be thought that inefficient statistics are being recommended for all situations. There are many cases where observations are very expensive, and obtaining a few more would entail great delay. Examples of this situation arise in agricultural experiments, where it often takes a season to get a set of observations, and where each observation is very expensive. In such cases the experimenters want to squeeze every drop of information out of their data. In these situations inefficient statistics would be uneconomical, and are not recommended.

A situation that often arises is that data are acquired in the natural course of administration of an organization. These data are filed away until the accumulation becomes mountainous. From time to time questions arise which can be answered by reference to the accumulated information. How much of these data will be used in the construction of say, estimates of parameters, depends on the precision desired for the answer. It will however often be less expensive to

get the desired precision by increasing the sample size by dipping deeper into the stock of data in the files, and using crude techniques of analysis, than to attain the required precision by restricting the sample size to the minimum necessary for use with "efficient" statistics.

It will often happen in other fields such as educational testing that it is less expensive to gather enough data to make the analysis by crude methods sufficiently precise, than to use the minimum sample sizes required by more refined methods. In some cases, as a result of the type of operation being carried out sample sizes are more than adequate for the purposes of estimation and testing significance. The experimenters have little interest in milking the last drop of information out of their data. Under these circumstances statistical workers would be glad to forsake the usual methods of analysis for rapid, inexpensive techniques that would offer adequate information, but for many problems such techniques are not available.

In the present paper several such techniques will be developed. For the most part we shall consider statistical methods which are applicable to estimating parameters. In a later paper we intend to consider some useful "inefficient" tests of significance.

2. Order statistics. If a sample $O_n = x_1', x_2', \cdots, x_n'$ of size n is drawn from a continuous probability density function $f(x)$, we may rearrange and renumber the observations within the sample so that

$$x_1 < x_2 < \cdots < x_n \tag{1}$$

(the occurrence of equalities is not considered because continuity implies zero probability for such events). The x_i's are sometimes called *order statistics*. On occasion we write $x(i)$ rather than x_i. Throughout this paper the use of primes on subscripted x's indicates that the observations are taken without regard to order, while unprimed subscripted x's indicate that the observations are order statistics satisfying (1). Similarly $x(n_i)$ will represent the n_ith order statistic, while $x'(n_i)$ would represent the n_ith observation, if the observations were numbered in some random order. The notation here is essentially the *opposite* of usual usage, in which attention is called to the order statistics by the device of primes or the introduction of a new letter. The present reversal of usage seems justified by the viewpoint of the article—that in the problems under consideration the use of order statistics is the natural procedure.

An example of a useful order statistic is the median; when $n = 2m + 1$ ($m = 0, 1, \cdots$), x_{m+1} is called the median and may be used to estimate the population median, i.e. u defined by

$$\int_{-\infty}^{u} f(t)\, dt = \tfrac{1}{2}.$$

In the case of symmetric distributions, the population mean coincides with u and x_{m+1} will be an unbiased estimate of it as well. When $n = 2m$ ($m = 1, 2,$

$\cdots$), the median is often defined as $\frac{1}{2}(x_m + x_{m+1})$. The median so defined is an unbiased estimate of the population median in the case of symmetric distributions; however for most asymmetric distributions $\frac{1}{2}(x_m + x_{m+1})$ will only be unbiased asymptotically, that is in the limit as n increases without bound. For another definition of the sample median see Jackson [8, 1921]. When x is distributed according to the normal distribution

$$N(x, a, \sigma^2) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(1/2\sigma^2)(x-a)^2},$$

the variance of the median is well known to tend to $\pi\sigma^2/2n$ as n increases.

It is doubtful whether we can accurately credit anyone with the introduction of the median. However for some of the results in the theory of order statistics it is easier to give credit. In this section we will restrict the discussion to the order statistics themselves, as opposed to the general class of statistics, such as the range $(x_n - x_1)$, which are derived from order statistics. We shall call the general class of statistics which are derived from order statistics, and use the value ordering (1) in their construction, *systematic statistics*.

The large sample distribution of extreme values (examples x_r, x_{n-s+1} for r, s fixed and $n \to \infty$) has been considered by Tippett [17, 1925] in connection with the range of samples drawn from normal populations; by Fisher and Tippett [3, 1928] in an attempt to close the gap between the limiting form of the distribution and results tabled by Tippett [17], by Gumbel [5, 1934] (and in many other papers, a large bibliography is available in [6, Gumbel 1939]), who dealt with the more general case $r \geq 1$, while the others mentioned considered the special case of $r = 1$; and by Smirnoff who considers the general case of x_r, in [15, 1935] and also [16] the limiting form of the joint distribution of x_r, x_s, for r and s fixed as $n \to \infty$.

In the present paper we shall not usually be concerned with the distribution of extreme values, but shall rather be considering the limiting form of the joint distribution of $x(n_1)$, $x(n_2)$, $\cdots$, $x(n_k)$, satisfying

CONDITION 1. $\lim_{n\to\infty} \frac{n_i}{n} = \lambda_i$; $\quad i = 1, 2, \cdots, k$;

$$\lambda_1 < \lambda_2 < \cdots < \lambda_k .$$

In other words the proportion of observations less than or equal to $x(n_i)$ tends to a fixed proportion which is bounded away from 0 and 1 as n increases. K. Pearson [13, 1920] supplies the information necessary to obtain the limiting distribution of $x(n_1)$, and limiting joint distribution of $x(n_1)$, $x(n_2)$. Smirnoff gives more rigorous derivations of the limiting form of the marginal distribution of the $x(n_i)$ [15, 1935] and the limiting form of the joint distribution of $x(n_i)$ and $x(n_j)$ [16] under rather general conditions. Kendall [10, 1943, pp. 211–14] gives a demonstration leading to the limiting form of the joint distribution.

Since we will be concerned with statements about the asymptotic properties of the distributions of certain statistics, it may be useful to include a short dis-

cussion of their implications both practical and theoretical. If we have a statistic $\hat{\theta}(O_n)$ based on a sample O_n: x_1', x_2', $\cdots$, x_n' drawn from a population with cumulative distribution function $F(x)$ it often happens that the function $(\hat{\theta} - \theta)/\sigma_n = y_n$, where σ_n is a function of n is such that

$$\text{(A)} \qquad \lim_{n\to\infty} P(y_n < t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-\frac{1}{2}x^2}\, dx.$$

When this condition (A) is satisfied we often say: $\hat{\theta}$ *is asymptotically normally distributed with mean* θ *and variance* σ_n^2. We will not be in error if we use the statement in italics provided we interpret it as synonymous with (A). However there are some pitfalls which must be avoided. In the first place condition (A) may be true even if the distribution function of y_n, or of $\hat{\theta}$, has no moments even of fractional orders for any n. Consequently we do not imply by the italicized statement that $\lim_{n\to\infty} E[\hat{\theta}(O_n)] = \theta$, nor that $\lim_{n\to\infty} \{[E(\hat{\theta}^2) - [E(\hat{\theta})]^2\} = \sigma_n^2$, for, as mentioned, these expressions need not exist for (A) to be true. Indeed we shall demonstrate that Condition (A) is satisfied for certain statistics even if their distribution functions are as momentless as the startling distributions constructed by Brown and Tukey [1, 1946]. Of course it may be the case that all moments of the distribution of $\hat{\theta}$ exist and converge as $n \to \infty$ to the moments of a normal distribution with mean θ and variance σ_n^2. Since this implies (A), but not conversely, this is a stronger convergence condition than (A). (See for example J. H. Curtiss [2, 1942].) However the important implication of (A) is that for sufficiently large n each percentage point of the distribution of $\hat{\theta}$ will be as close as we please to the value which we would compute from a normal distribution with mean θ and variance σ_n^2, independent of whether the distribution of $\hat{\theta}$ has these moments or not.

Similarly if we have several statistics $\hat{\theta}_1$, $\hat{\theta}_2$, $\cdots$, $\hat{\theta}_k$, each depending upon the sample O_n : x_1', x_2', $\cdots$, x_n', we shall say *that the* $\hat{\theta}_i$ *are asymptotically jointly normally distributed with means* θ_i, *variances* $\sigma_i^2(n)$, *and covariances* $\rho_{ij}\sigma_i\sigma_j$, when

$$\text{(B)} \qquad \lim_{n\to\infty} P(y_1 < t_1, y_2 < t_2, \cdots, y_k < t_k) = K \int_{-\infty}^{t_1} \int_{-\infty}^{t_2} \cdots \int_{-\infty}^{t_k} e^{-\frac{1}{2}Q^2}\, dx_1\, dx_2 \cdots dx_k,$$

where $y_i = (\hat{\theta}_i - \theta_i)/\sigma_i$, and Q^2 is the quadratic form associated with a set of k jointly normally distributed variables with variances unity and covariances ρ_{ij}, and K is a normalizing constant. Once again the statistics $\hat{\theta}_i$ may not have moments or product moments, the point that interests us is that the probability that the point with coordinates $(\hat{\theta}_1, \hat{\theta}_2, \cdots, \hat{\theta}_k)$ falls in a certain region in a k-dimensional space can be given as accurately as we please for sufficiently large samples by the right side of (B).

Since the practicing statistician is very often really interested in the probability that a point will fall in a particular region, rather than in the variance

or standard deviation of the distribution itself, the concepts of asymptotic normality given in (A) and (B) will usually not have unfortunate consequences. For example, the practicing statistician will usually be grateful that the sample size can be made sufficiently large that the probability of a statistic falling into a certain small interval can be made as near unity as he pleases, and will not usually be concerned with the fact that, say, the variance of the statistic may be unbounded.

Of course, a very real question may arise: how large must n be so that the probability of a statistic falling within a particular interval can be sufficiently closely approximated by the asymptotic formulas? If in any particular case the sample size must be ridiculously large, asymptotic theory loses much of its practical value. However for statistics of the type we shall usually discuss, computation has indicated that in many cases the asymptotic theory holds very well for quite small samples.

For the demonstration of the joint asymptotic normality of several order statistics we shall use the following two lemmas.

LEMMA 1. *If a random variable $\hat{\theta}(O_n)$ is asymptotically normally distributed converging stochastically to θ, and has asymptotic variance $\sigma^2(n) \underset{n\to\infty}{\to} 0$, where n is the size of the sample $O_n : x_1', x_2', \cdots, x_n'$, drawn from the probability density function $h(x)$, and $g(\hat{\theta})$ is a single-valued function with a nonvanishing continuous derivative $g'(\hat{\theta})$ in the neighborhood of $\hat{\theta} = \theta$, then $g(\hat{\theta})$ is asymptotically normally distributed converging stochastically to $g(\theta)$ with asymptotic variance $\sigma_n^2[g'(\theta)]^2$.*

PROOF. By the conditions of the lemma

$$\lim_{n\to\infty} P\left[\frac{\hat{\theta} - \theta}{\sigma_n} < t\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-\frac{1}{2}u^2}\, du.$$

Now if $t\sigma_n = \Delta\theta$, $\Delta\theta = \hat{\theta} - \theta$, using the mean value theorem there is a θ_1 in the interval $[\theta, \hat{\theta}]$, such that

$$g(\hat{\theta}) = g(\theta) + (\hat{\theta} - \theta)g'(\theta_1),$$

which implies

$$\lim_{n\to\infty} P\left(\frac{\hat{\theta} - \theta}{\sigma_n} < t\right) = \lim_{n\to\infty} P\left(\frac{g(\hat{\theta}) - g(\theta)}{\sigma_n g'(\theta_1)} < t\right), \qquad g'(\theta_1) \neq 0,$$

where θ_1 is a function of n. However $\lim_{\Delta\theta\to 0} g'(\theta_1) = g'(\theta)$ so we may write

$$\lim_{n\to\infty} P\left(\frac{\hat{\theta} - \theta}{\sigma_n} < t\right) = \lim_{n\to\infty} P\left(\frac{g(\hat{\theta}) - g(\theta)}{\sigma_n g'(\theta)} < t\right), \qquad g'(\theta) \neq 0.$$

where the form of the expression on the right is the one required to complete the proof of the lemma.

Of course if we have several random variables $\hat{\theta}_1, \hat{\theta}_2, \cdots, \hat{\theta}_k$, we can prove by an almost identical argument that

LEMMA 2. *If the random variables $\hat{\theta}_i(O_n)$ are asymptotically jointly normally*

distributed converging stochastically to θ_i, *and have asymptotic variances* $\sigma_i^2(n) \underset{n\to\infty}{\longrightarrow} 0$, *and covariances* $\rho_{ij}\sigma_i\sigma_j$, *where* n *is the size of the sample* $O_n : x_1', x_2', \cdots, x_n'$ *drawn from the probability density function* $h(x)$, *and* $g_i(\hat{\theta}_i)$, $i = 1, 2, \cdots, k$, *are single-valued functions with nonvanishing continuous derivatives* $g_i'(\hat{\theta}_i)$ *in the neighborhood of* $\hat{\theta}_i = \theta_i$, *then the* $g_i(\hat{\theta}_i)$ *are jointly asymptotically normally distributed with means* $g_i(\theta_i)$, *variances* $\sigma_i^2[g_i'(\theta_i)]^2$ *and covariances* $\rho_{ij}\sigma_i\sigma_j g_i'(\theta_i)g_j'(\theta_j)$.

The following condition represents restrictions on the probability density function $f(x)$ sufficient for the derivation of the limiting form of the joint distribution of the $x(n_i)$ satisfying Condition 1.

CONDITION 2. *The probability density function* $f(x)$ *is continuous, and does not vanish in the neighborhood of* u_i, *where*

$$\int_{-\infty}^{u_i} f(x)\,dx = \lambda_i, \qquad i = 1, 2, \cdots, k.$$

If we recall the discussion of condition (B) above, the theorem of Pearson and Smirnoff may be stated:

THEOREM 1. *If a sample* $O_n : x_1, x_2, \cdots, x_n$ *is drawn from* $f(x)$ *satisfying Condition 2, and if* $x(n_1)$, $x(n_2)$ *satisfy Condition 1 as* $n \to \infty$, *then* $x(n_1)$, $x(n_2)$ *are asymptotically distributed according to the normal bivariate distribution with means* u_1, u_2,

$$\int_{-\infty}^{u_i} f(x)\,dx = \lambda_i,$$

and variances

$$\sigma_i^2 = \frac{\lambda_i(1-\lambda_i)}{n[f(u_i)]^2}, \qquad i = 1, 2,$$

and covariance

$$\rho_{12}\sigma_1\sigma_2 = \frac{\lambda_1(1-\lambda_2)}{nf(u_1)f(u_2)}.$$

Theorem 1 has an obvious generalization which seems not to have been carried out in the literature. The generalization may be stated:

THEOREM 2. *If a sample* $O_n : x_1, x_2, \cdots, x_n$ *is drawn from* $f(x)$ *satisfying Condition 2, and if* $x(n_1), x(n_2), \cdots, x(n_k)$ *satisfy Condition 1 as* $n \to \infty$, *then the* $x(n_i)$, $i = 1, 2, \cdots, k$, *are asymptotically distributed according to the normal multivariate distribution, with means* u_i,

$$\int_{-\infty}^{u_i} f(x)\,dx = \lambda_i,$$

and variances

$$\sigma_i^2 = \frac{\lambda_i(1-\lambda_i)}{nf(u_i)^2}, \qquad i = 1, 2, \cdots, k,$$

and covariances

$$\rho_{ij}\sigma_i\sigma_j = \frac{\lambda_i(1-\lambda_j)}{nf(u_i)f(u_j)}, \qquad 1 \le i < j \le k.$$

PROOF. We shall carry out the demonstration for the uniform distribution

$$f(x) = \begin{cases} 1, & 0 \le x \le 1, \\ 0, & \text{elsewhere,} \end{cases}$$

and then utilize the fact that by a suitable transformation of the uniform distribution we may get any $f(x)$ satisfying Condition 2. Of course for the particular case of the uniform distribution all moments of the $x(n_i)$ exist and converge to those of the asymptotic theory.

The joint probability density of the $x(n_i)$, satisfying Condition 1 and drawn from $f(x)$, is given by

$$g[x(n_1), x(n_2), \cdots, x(n_k)] = \frac{n!}{(n_1-1)!(n-n_k)!\prod_{i=2}^{k}(n_i-n_{i-1}-1)!} \left(\int_0^{x(n_1)} dt_1\right)^{n_1-1}\left(\int_{x(n_k)}^{1} dt_{k+1}\right)^{n-n_k}\prod_{i=2}^{k}\left[\int_{x(n_{i-1})}^{x(n_i)} dt_i\right]^{n_i-n_{i-1}-1}. \tag{2}$$

Performing the indicated integrations we get from the right of (2)

$$Cx(n_1)^{n_1-1}\prod_{i=2}^{k}[x(n_i)-x(n_{i-1})]^{n_i-n_{i-1}-1}[1-x(n_k)]^{n-n_k}, \tag{3}$$

where C is the multinomial coefficient on the right of (2). It is well known that for the uniform distribution $E[x(n_i)] = \frac{n_i}{n+1}$, or asymptotically $\frac{n_i}{n}$, $i = 1, 2, \cdots, k$. We make the transformation $y_i = \left(x(n_i) - \frac{n_i}{n}\right)\sqrt{n}$, leading to

$$C_1\left(\frac{n_1}{n}+\frac{y_1}{\sqrt{n}}\right)^{n_1-1}\prod_{i=2}^{k}\left(\frac{n_i-n_{i-1}}{n}+\frac{[y_i-y_{i-1}]}{\sqrt{n}}\right)^{n_i-n_{i-1}-1} \cdot\left(\frac{n-n_k}{n}-\frac{y_k}{\sqrt{n}}\right)^{n-n_k}. \tag{4}$$

Using the usual technique of factoring out expressions like

$$\left(\frac{n_i-n_{i-1}}{n}\right)^{n_i-n_{i-1}-1},$$

we rewrite (4) with C_2 as a new constant, and setting $\lambda_i = \frac{n_i}{n}$

$$C_2\left(1+\frac{y_1}{\lambda_1\sqrt{n}}\right)^{n_1-1} \cdot\prod_{i=2}^{k}\left(1+\frac{(y_i-y_{i-1})}{(\lambda_i-\lambda_{i-1})\sqrt{n}}\right)^{n_i-n_{i-1}-1}\left(1-\frac{y_k}{(1-\lambda_k)\sqrt{n}}\right)^{n-n_k}. \tag{5}$$

Now taking the logarithm of (5), expanding, neglecting terms $O\left(\frac{1}{\sqrt{n}}\right)$ and higher, collecting terms and taking the antilogarithm we get the approximate asymptotic distribution of the order statistics

$$g(x(n_1), x(n_2), \cdots, x(n_k))$$

$$= C_3 \exp\left[-\frac{1}{2}\left\{\sum_{i=1}^{k} y_i^2 \frac{\lambda_{i+1} - \lambda_{i-1}}{(\lambda_{i+1} - \lambda_i)(\lambda_i - \lambda_{i-1})} - 2\sum_{i=2}^{k} \frac{y_i y_{i-1}}{\lambda_i - \lambda_{i-1}}\right\}\right], \tag{6}$$

where $\lambda_0 = 0$, $\lambda_{k+1} = 1$. Now setting up the matrix of the coefficients of the quadratic expression in the exponent

$$A_{ii} = \frac{\lambda_{i+1} - \lambda_{i-1}}{(\lambda_{i+1} - \lambda_i)(\lambda_i - \lambda_{i-1})}; \qquad A_{i,i-1} = A_{i-1,i} = -\frac{1}{\lambda_i - \lambda_{i-1}},$$

$i = 1, 2, \cdots, k$; $A_{ij} = 0$, $|\, i - j \,| > 1$. To obtain the variances and covariances we need

$$A^{ij} = \frac{\text{cofactor of } A_{ij} \text{ in } ||\, A_{ij} \,||}{\text{determinant } A_{ij}}$$

(see for example Wilks [18, p. 63 et seq.]). Now

$$|\, A \,| = \text{determinant } A_{ij} = \prod_{1}^{k+1} \frac{1}{\lambda_i - \lambda_{i-1}}; \tag{7}$$

$$\text{cofactor of } A_{ii} = \lambda_i(1 - \lambda_i)\,|\, A \,|,\ i = 1, 2, \cdots, k.$$

$$\text{cofactor of } A_{ij} = \begin{cases} \lambda_i(1 - \lambda_j)\,|\, A \,|, & i < j \\ \lambda_j(1 - \lambda_i)\,|\, A \,|, & j < i. \end{cases}$$

This completes the proof for the uniform distribution.

If the uniform distribution is transformed into a probability density function $f(x)$ satisfying Condition 2, by an order preserving transformation, we appeal to Lemma 2. We notice that the $x(n_i)$ are transformed into $g[x(n_i)]$, and that the probability that $x(n_i)$ falls in the interval $[u_i, u_i + \Delta u_i]$ is transformed into the probability that $g[x(n_i)]$ falls in the interval $[g(u_i), g(u_i + \Delta u_i)]$. Using the mean value theorem we may write

$$g(u_i + \Delta u_i) = g(u_i) + \Delta u_i g'(u_i'),$$

where u_i' lies in the interval $[u_i, u_i + \Delta u_i]$. However

$$\lim_{\Delta u_i \to 0} g'(u_i') = g'(u_i).$$

The density for the uniform distribution in the interval $[u_i, u_i + \Delta u_i]$ is just Δu_i, and this same density will tend to $f(u_i)\Delta u_i g'(u_i)$. Therefore $g'(u_i) = 1/f(u_i)$, which completes the proof of Theorem 2.

It would often be useful to know the small sample distribution of the order statistics, particularly in the case where the sample is drawn from a normal.

Fisher and Yates' tables [4] give the expected values of the order statistics up to samples of size 50. However it would be very useful in the development of certain small sample statistics to have further information. It is perhaps too much to expect tabulated distribution functions, but at least the variances and covariances would be useful. A joint effort has resulted in the calculation for samples $n = 2, 3, \cdots, 10$ of the expected values to five decimal places, the variances to four decimal places, and the covariances to nearly two decimal places. It is expected that these tables will be published shortly.

3. Estimates of the mean of a normal distribution. It will be important in what follows to define efficiency and to indicate its interpretation. Then we shall construct some estimates of the means of certain distributions and compute their efficiencies. Except for the tables given, the discussion is applicable to the estimation of the mean of any symmetric distribution; and, of course, the concept of efficiency is still more general in its application. A statistic $\hat{\theta}(O_n)$, where O_n is the sample, is said to be an *efficient* estimate of θ if

i) $\sqrt{n}\,(\hat{\theta} - \theta)$ is asymptotically normally distributed with zero mean and finite variance, $\sigma^2(\hat{\theta})$, and

ii) for any other statistic $\hat{\theta}'$ with $\sqrt{n}(\hat{\theta}' - \theta)$ asymptotically normally distributed with zero mean and variance $\sigma^2(\hat{\theta}')$, $\sigma^2(\hat{\theta}) \leq \sigma^2(\hat{\theta}')$.

The ratio $\sigma^2(\hat{\theta})/\sigma^2(\hat{\theta}')$ is termed the efficiency of $\hat{\theta}'$ if $\hat{\theta}$ is an efficient estimate of θ. For discussion see Wilks [18, 1943]. The concepts of efficient statistic or estimate and of efficiency were introduced by R. A. Fisher. They serve as one measure of the amount of information a statistic draws from a sample. It is also common practice to speak of relative efficiencies, for example, of the statistics $\hat{\theta}'$ and $\hat{\theta}''$ described in ii) above, we say if $\sigma^2(\hat{\theta}') < \sigma^2(\hat{\theta}'')$ that the efficiency of $\hat{\theta}''$ relative to $\hat{\theta}'$ is the ratio of the smaller variance to the larger. This concept of efficiency has sometimes been used when the normality assumption has been violated by one or both statistics, when one or both are biased, and when small samples are considered. When used under these conditions the concept of efficiency becomes more difficult to interpret, although a comparison of the variation of two statistics about the value they are commonly estimating is often of value.

In the case of estimates of the mean a of a variable which is normally distributed according to $N(x, a, \sigma^2)$ from a sample of n, we can often express the variance of an asymptotically unbiased estimate as $\sigma^2(\hat{\theta}_i) = k_i\sigma^2/n$. The sample mean $\hat{\theta} = \Sigma x_i'/n$ is an efficient estimate of a with variance σ^2/n. Then in such cases the efficiency of $\hat{\theta}_i$ in estimating a is $1/k_i$. The interpretation is merely that to obtain the same precision using $\hat{\theta}_i$ as is possible with $\hat{\theta}$, one must use a sample k_i times as large.

Bearing in mind that we are at present searching for economical methods for analyzing large samples, it is clear that the concept of efficiency offers us a practical way of comparing cost of information with cost of obtaining it.

In the present section and in sections 4 and 5 we shall develop certain systematic estimates of parameters of normally distributed variables. Our procedure then will be to compare the efficiency of the systematic estimates with the efficient statistic for estimating the parameter in question, and also in sections 4 and 5 we compare our estimates with a statistic not involving squares or products. Of course the efficient statistic for estimating the mean of a normal is the sample mean, therefore in this section we will only compare our estimates with the sample mean.

We can construct unbiased estimates of the mean of a normal distribution from linear combinations of suitably chosen order statistics. These systematic statistics will be asymptotically normally distributed if the order statistics from which they are derived satisfy Condition 1. We will restrict ourselves to a useful practical case where equal weights are used. In other words the estimate discussed is just the average of k order statistics $k^{-1}\Sigma x(n_i)$. Suppose $x(n_i)$, $i = 1, 2, \cdots, k$ satisfy Condition 1, that $E[x(n_i)] = E[x(n_{k-i+1})]$, so that $E[\Sigma x(n_i)] = a$. An important unsolved question is to discover what spacing of the $x(n_i)$ will yield minimum variance, and thereafter at what rate does the efficiency of this optimumly spaced estimate increase with k. Computational methods bog down rapidly after $k = 3$. Because so little is known about this problem it seems worthwhile to offer some results for three arbitrary spacings (these results are of course useful in analyzing data).

If the $x(n_i)$ satisfy Theorem 2 we may approximate the variance of the systematic statistic $\hat{\theta}_k = \Sigma x(n_i)/k$ by the usual formula

$$\sigma^2(\hat{\theta}_k) = E[\Sigma x(n_i)/k]^2 - [E(\Sigma x(n_i)/k)]^2. \tag{8}$$

We lose no generality by assuming the mean and variance of the underlying normal to be 0 and 1 respectively. Then using the fact that $\Sigma u_i = 0$, and the result of Theorem 1 we rewrite (8) as

$$\sigma^2(\hat{\theta}_k) = E[\Sigma(x(n_i) - u_i)/k]^2 = \frac{1}{k^2 n}\left[\sum_{i=1}^{k} \frac{\lambda_i(1-\lambda_i)}{f_i^2} + 2\sum_{i<j} \frac{\lambda_i(1-\lambda_j)}{f_i f_j}\right], \tag{9}$$

where $f_m = f(u_m)$.

Using the symmetry which makes $\lambda_i = 1 - \lambda_{k-i+1}$, $f_i = f_{k-i+1}$, and the fact that for $k = 2r + 1$, $f_{r+1} = 1/\sqrt{2\pi}$, $\lambda_{r+1} = \frac{1}{2}$, we may simplify the right side of equation (9) with the following results for $k = 1, 2, \cdots, 7$. The factor $1/k^2$ has not been disturbed. We also write the general formulas for the simplified form of (9), but we omit a rather lengthy combinatorial argument which establishes the generalization.

$k = 1$: $\dfrac{\pi}{2n}$

$k = 2$: $\dfrac{2\lambda_1}{4nf_1^2}$

$$
\begin{aligned}
k = 3&: \frac{2}{9n}\left[\frac{\lambda_1}{f_1^2} + \frac{\lambda_1\sqrt{2\pi}}{f_1} + \frac{\pi}{4}\right] \\
k = 4&: \frac{2}{16n}\left[\frac{\lambda_1}{f_1^2} + \frac{2\lambda_1}{f_1 f_2} + \frac{\lambda_2}{f_2^2}\right] \\
k = 5&: \frac{2}{25n}\left[\frac{\lambda_1}{f_1^2} + \frac{2\lambda_1}{f_1 f_2} + \frac{\lambda_2}{f_2^2} + \sqrt{2\pi}\left(\frac{\lambda_1}{f_1} + \frac{\lambda_2}{f_2}\right) + \frac{\pi}{4}\right] \\
k = 6&: \frac{2}{36n}\left[\frac{\lambda_1}{f_1^2} + \frac{\lambda_2}{f_2^2} + \frac{\lambda_3}{f_3^2} + \frac{2\lambda_1}{f_1 f_2} + \frac{2\lambda_1}{f_1 f_3} + \frac{2\lambda_2}{f_2 f_3}\right] \\
k = 7&: \frac{2}{49n}\left[\frac{\lambda_1}{f_1^2} + \frac{\lambda_2}{f_2^2} + \frac{\lambda_3}{f_3^2} + \frac{2\lambda_1}{f_1 f_2} + \frac{2\lambda_1}{f_1 f_3} + \frac{2\lambda_2}{f_2 f_3}\right. \\
&\qquad\qquad \left. + \sqrt{2\pi}\left(\frac{\lambda_1}{f_1} + \frac{\lambda_2}{f_2} + \frac{\lambda_3}{f_3}\right) + \frac{\pi}{4}\right] \\
k = 2r&: \frac{2}{(2r)^2 n}\left[\sum_{i=1}^{r}\frac{\lambda_i}{f_i^2} + 2\sum_{1\le i<j\le r}\frac{\lambda_i}{f_i f_j}\right], \qquad r \ge 1 \\
k = 2r+1&: \frac{2}{(2r+1)^2 n}\left[\frac{(2r)^2 n}{2}\,\sigma^2(\hat{\theta}_{2r}) + \sqrt{2\pi}\sum_{i=1}^{r}\frac{\lambda_i}{f_i} + \frac{\pi}{4}\right], \qquad r \ge 1.
\end{aligned}
\tag{10}
$$

In addition to the possibility of minimizing the equations of (10) by numerical methods, three other procedures suggest themselves: i) to space the order statistics uniformly in probability; ii) to choose those k order statistics whose expected values are equal to the expected values of the order statistics in a sample of size k drawn from a unit normal; iii) to choose $\lambda_i = (i - \frac{1}{2})/k$. The following table lists for $k = 1$, 2, and 3 the expected values u_i of the order statistics and the probability to the left of the expected values λ_i for each of the procedures. The chosen order statistics are counted from left to right. It will be noticed that the third method gives very good results, and has the value of simplicity of formula. The following table gives a comparison between the efficiencies resulting from spacing by the three methods. The three optimum cases are included for completeness.

Statisticians planning to use the method of expected values suggested above will find Fisher and Yates [4, 1943] table of the expected values of the order statistics in samples of size k drawn from a unit normal helpful for computing the λ_i. Alternatively the following table of λ_i might be used.

As an example of the use of Table III, suppose we are using the expected value method for estimating the mean of a large sample drawn from a normal distribution $N(x, a, \sigma^2)$. If we are willing to use 6 observations out of 1000 for this purpose Table III indicates the selection of x_{103}, x_{261}, x_{421}, x_{580}, x_{740}, x_{898}. Furthermore Table II indicates that the variance of the estimate of a based on the average of these six observations will be approximately $\sigma^2/.948n$, $n = 1000$.

4. Estimates of the standard deviation. The statistic

$$s^2 = \sum_{i=1}^{n} (x_i' - \bar{x})^2/(n-1),$$

TABLE I

Comparison of the order statistics which would be chosen according to each of the four procedures for subsamples of $k = 1, 2, 3$

k	Order Statistic	Optimum		Equal Probability		Expected Values		$\lambda_i=(i-\frac{1}{2})/k$	
		u_i	λ_i	u_i	λ_i	u_i	λ_i	u_i	λ_i
1	First	.0000	.5000	.0000	.5000	.0000	.5000	.0000	.5000
2	First	−.6121	.2702	−.4307	.3333	−.5642	.2863	−.6745	.2500
	Second	.6121	.7298	.4307	.6667	.5642	.7137	.6745	.7500
3	First	−.9056	.1826	−.6745	.2500	−.8463	.1967	−.9674	.1667
	Second	.0000	.5000	.0000	.5000	.0000	.5000	.0000	.5000
	Third	.9056	.8174	.6745	.7500	.8463	.8033	.9674	.8333

TABLE II

Comparison of the efficiencies of four methods of spacing k order statistics used in the construction of an estimate of the mean

k	$\lambda_i=i/(k+1)$	Expected Values*	$\lambda_i=(i-\frac{1}{2})/k$	Optimum
1	.637	.637	.637	.637
2	.793	.809	.808	.810
3	.860	.878	.878	.879
4	.896	.914	.913	
5	.918	.933	.934	
6	.933	.948	.948	
7	.944	.956	.957	
8	.952	.963	.963	
9	.957	.968	.969	
10	.962	.972	.973	

* The u_i are chosen equal to the expected values of the order statistics of a sample of size k.

where $\bar{x} = \sum_{i=1}^{n} x_i'/n$ is well known to be an unbiased estimate of the population variance σ^2, for $n > 1$. However s is not in general an unbiased estimate of σ. We are not interested here in the question of when we should estimate σ and when it is more advantageous to estimate σ^2. All we want is to have an

unbiased estimate of σ, based on sums of squares, to compare with another unbiased estimate based on order statistics. In the case of observations drawn from a normal distribution

$$s' = \frac{(\frac{1}{2}n)^{\frac{1}{2}}\Gamma(\frac{1}{2}[n-1])}{\Gamma(\frac{1}{2}n)} \sqrt{\frac{\Sigma(x_i' - \bar{x})^2}{n}}, \tag{11}$$

is an unbiased estimate of σ (see for example Kenney [11], with variance

$$\sigma^2(s') = \left\{\tfrac{1}{2}\left[\frac{\Gamma(\frac{1}{2}[n-1])}{\Gamma(\frac{1}{2}n)}\right]^2 (n-1) - 1\right\}\sigma^2. \tag{12}$$

TABLE III*

$P(x < u_{i|k}) \times 10^4$, $u_{i|k} = E(x_{i|k})$, $x_{i|k}$ is the ith order statistic in a sample of size k drawn from a normal distribution $N(x, 0, 1)$

k \ i	1	2	3	4	5	6	7	8	9	10
1	5000									
2	2863	7137								
3	1987	5000	8013							
4	1516	3832	6168	8484						
5	1224	3103	5000	6897	8776					
6	1025	2605	4201	5799	7395	8975				
7	0881	2244	3622	5000	6378	7756	9119			
8	0773	1971	3182	4394	5606	6818	8030	9227		
9	0688	1756	2837	3919	5000	6082	7163	8244	9312	
10	0619	1584	2559	3536	4512	5488	6464	7441	7416	9381

* The table is given to more places than necessary for the purpose suggested because it may be of interest in other applications. The $E(x_{i|k})$ from which the table was derived were computed to five decimal places.

For most practical purposes however, when $n > 10$, the bias in s is negligible. For large samples $\sigma^2(s')$ approaches $\sigma^2/2n$.

4A. The range as an estimate of σ. As mentioned in the Introduction, section 1, it is now common practice in industry to estimate the standard deviation by means of a multiple of the range $R' = c_n(x_n - x_1)$, for small samples, where $c_n = 1/[E(y_n) - E(y_1)]$, y_n and y_1 being the greatest and least observations drawn from a sample of size n from a normal distribution $N(y, a, 1)$. Although we are principally interested in large sample statistics, for the sake of completeness, we shall include a few remarks about the use of the range in small samples.

Now R' is an unbiased estimate of σ, and its variance may be computed for small samples, see for example Hartley [7, 1942]. In the present case, although both R' and s' are unbiased estimates of σ, they are not normally distributed,

nor are we considering their asymptotic properties; therefore the previously defined concept of efficiency does not apply. We may however use the ratio of the variances as an arbitrary measure of the relative precision of the two statistics. The following table lists the ratio of the variances of the two statistics, as well as the variances themselves expressed as a multiple of the population variance for samples of size $n = 2, 3, \cdots, 10$.

4B. Quasi ranges for estimating σ. The fact that the ratio $\sigma^2(s')/\sigma^2(R')$ falls off in Table IV as n increases makes it reasonable to inquire whether it might not be worthwhile to change the systematic estimate slightly by using the statistic $c_{1|n}[x_{n-1} - x_2]$, or more generally $c_{r|n}[x_{n-r} - x_{r+1}]$ where $c_{r|n}$ is the multiplicative constant which makes the expression an unbiased estimate of σ (in particular $c_{r|n}$ is the constant to be used when we count in $r + 1$ observations from each end of a sample of size n, thus $c_{r|n} = 1/[E(y_{n-r} - y_{r+1})]$ where the

TABLE IV

Relative precision of s' and R', and their variances expressed as a multiple of σ^2, the population variance

n	$\sigma^2(s')/\sigma^2(R')$	$\sigma^2(s')/\sigma^2$	$\sigma^2(R')/\sigma^2$
2	1.000	.570	.570
3	.990	.273	.276
4	.977	.178	.182
5	.962	.132	.137
6	.932	.104	.112
7	.910	.0864	.0949
8	.889	.0738	.0830
9	.869	.0643	.0740
10	.851	.0570	.0670

y's are drawn from $N(y, a, 1)$). This is certainly the case for large values of n, but with the aid of the unpublished tables mentioned at the close of section 2, we can say that it seems not to be advantageous to use $c_{1|n}[x_{n-1} - x_2]$ for $n \leq 10$. Indeed the variance $c_{1|10}[x_9 - x_2]$, for the unit normal seems to be about .10, as compared with $\sigma^2(R')/\sigma^2 = .067$ as given by Table IV, for $n = 10$. The uncertainty in the above statements is due to a question of significant figures.

Considerations which suggest constructing a statistic based on the difference of two order statistics which are not extreme values in small samples, weigh even more heavily in large samples. A reasonable estimate of σ for normal distributions, which could be calculated rapidly by means of punched-card equipment is

$$\hat{\sigma} = \frac{1}{c}[x(n_2) - x(n_1)], \tag{13}$$

where the $x(n_i)$ satisfy Condition 1, and where $c = u_2 - u_1$, u_2 and u_1 are the expected values of the n_2 and n_1 order statistics of a sample of size n drawn from a unit normal. Without loss of generality we shall assume the x_i are drawn from a unit normal. Furthermore we let $\frac{n_2}{n} = \lambda_2 = 1 - \lambda_1 = 1 - \frac{n_1}{n}$. Of course σ will be asymptotically normally distributed, with variance

$$\sigma^2(\hat{\sigma}) = \frac{2}{nc^2}\left[\frac{\lambda_1(1-\lambda_1)}{[f(u_1)]^2} + \frac{\lambda_2(1-\lambda^2)}{[f(u_2)]^2} - \frac{2\lambda_1(1-\lambda_2)}{f(u_1)f(u_2)}\right]. \tag{14}$$

Because of symmetry $f(u_1) = f(u_2)$; using this and the fact that $\lambda_1 = 1 - \lambda_2$, we can reduce (14) to

$$\sigma^2(\hat{\sigma}) = \frac{2}{nc^2}\frac{\lambda_1(1-2\lambda_1)}{[f(u_1)]^2}. \tag{15}$$

We are interested in optimum spacing in the minimum variance sense. The minimum for $\sigma^2(\hat{\sigma})$ occurs when $\lambda_1 \doteq .0694$, and for that value of λ_1, $\sigma^2(\hat{\sigma}) \doteq .767\ \sigma^2/n$. Asymptotically s' is also normally distributed, with $\sigma^2(s') = \sigma^2/2n$. Therefore we may speak of the efficiency of $\hat{\sigma}$ as an estimate of σ as .652. It is useful to know that the graph of $\sigma^2(\hat{\sigma})$ is very flat in the neighborhood of the minimum, and therefore varying λ_1 by .01 or .02 will make little difference in the efficiency of the estimate $\hat{\sigma}$ (providing of course that c is appropriately adjusted). K. Pearson [13] suggested this estimate in 1920. It is amazing that with punched-card equipment available it is practically never used when the appropriate conditions described in the Introduction are present.

The occasionally used semi-interquartile range, defined by $\lambda_1 = .25$ has an efficiency of only .37 and an efficiency relative to $\hat{\sigma}$ of only .56.

As in the case of the estimate of the mean by systematic statistics, it is pertinent to inquire what advantage may be gained by using more order statistics in the construction of the estimate of σ. If we construct an estimate based on four order statistics, and then minimize the variance, it is clear that the extreme pair of observations will be pushed still further out into the tails of the distribution. This is unsatisfactory from two points of view in practice: i) we will not actually have an infinite number of observations, therefore the approximation concerning the normality of the order statistics may not be adequate if λ_1 is too small, even in the presence of truly normal data; ii) the distribution functions met in practice often do not satisfy the required assumption of normality, although over the central portion of the function containing most of the probability, say except for the 5% in each tail normality may be a good approximation. In view of these two points it seems preferable to change the question slightly and ask what advantage will accrue from holding two observations at the optimum values just discussed (say $\lambda_1 = .07$, $\lambda_2 = .93$) and introducing two additional observations more centrally located.

We define a new statistic

$$\hat{\sigma}' = \frac{1}{c'}[x(n_4) + x(n_3) - x(n_2) - x(n_1)], \tag{16}$$

$c' = E[x(n_4) + x(n_3) - x(n_2) - x(n_1)]$, where the observations are drawn from a unit normal. We take $\lambda_1 = 1 - \lambda_4$, $\lambda_2 = 1 - \lambda_3$, $\lambda_1 = .07$. It turns out that $\sigma^2(\hat{\sigma}')$ is minimized for λ_2 in the neighborhood of .20, and that the efficiency compared with s' is a little more than .75. Thus an increase of two observations in the construction of our estimate of σ increases the efficiency from .65 to .75. We get practically the same result for $.16 \leq \lambda_2 \leq .22$.

Furthermore, it turns out that using $\lambda_1 = .02$, $\lambda_2 = .08$, $\lambda_3 = .15$, $\lambda_4 = .25$, $\lambda_5 = .75$, $\lambda_6 = .85$, $\lambda_7 = .92$, $\lambda_8 = .98$, one can get an estimate of σ based on eight order statistics which has an efficiency of .896. This estimate is more efficient than either the mean deviation about the mean or median for estimating σ. The estimate is of course

$$\hat{\sigma}'' = [x(n_8) + x(n_7) + x(n_6) + x(n_5) - x(n_4) - x(n_3) - x(n_2) - x(n_1)]/C,$$

where $C = 10.34$.

To summarize: in estimating the standard deviation σ of a normal distribution from a large sample of size n, an unbiased estimate of σ is

$$\hat{\sigma} = \frac{1}{c}(x_{n-r+1} - x_r),$$

where $c = E(y_{n-r+1} - y_r)$ where the y's are drawn from $N(y, a, 1)$. The estimate $\hat{\sigma}$ is asymptotically normally distributed with variance

$$\sigma^2(\hat{\sigma}) = \frac{2}{nc^2}\frac{\lambda_1(1 - 2\lambda_1)}{[f(u_1)]^2},$$

where $\lambda_1 = r/n$, $f(u_1) = N(E(x_r), 0, \sigma^2)$. We minimize $\sigma^2(\hat{\sigma})$ for large samples when $\lambda_1 \doteq .0694$, and for that value of λ_1,

$$\sigma^2_{\text{opt}}(\hat{\sigma}) \doteq \frac{.767\sigma^2}{n}.$$

The unbiased estimate of σ

$$\hat{\sigma}' = \frac{1}{c'}(x_{n-r+1} + x_{n-s+1} - x_s - x_r)$$

may be used in lieu of $\hat{\sigma}$. If $\lambda_1 = r/n$, $\lambda_2 = s/n$ we find

$$\sigma^2(\hat{\sigma}' \mid \lambda_1 = .07, \lambda_2 = .20) \doteq \frac{.66\sigma^2}{n}.$$

4C. The mean deviations about the mean and median. The next level of computational difficulty we might consider for the construction of an estimate of σ is the process of addition. The mean deviation about the mean is a well known, but not often used statistic. It is defined by

$$\text{m.d.} = \sum_{i=1}^{n} | x'_i - \bar{x} | /n. \tag{17}$$

For large samples from a normal distribution the expected value of m.d. is $\sqrt{\frac{2}{\pi}}\sigma$, therefore to obtain an unbiased estimate of σ we define the new statistic $A = \sqrt{\frac{\pi}{2}}$ m.d. Now for large samples A has variance $\sigma^2[\frac{1}{2}(\pi - 2)]/n$, or an efficiency of .884. However there are slight awkwardnesses in the computation of A which the mean deviation about the median does not have.

It turns out that for samples of size $n = 2m + 1$ drawn from a normal distribution $N(y, a, 0)$ the statistic

$$M' = \sqrt{\frac{\pi}{2}} \frac{\sum |x_i - x_{m+1}|}{2m} \tag{18}$$

asymptotically has mean σ and variance

$$\sigma^2(M') = \frac{1}{2m}\left(\frac{\pi - 2}{2}\right)\sigma^2. \tag{19}$$

Thus in estimating the standard deviation of a normal distribution from large samples we can get an efficiency of .65 by the judicious selection of two observations from the sample, an efficiency of .75 by using four observations, and an efficiency of .88 by using the mean deviation of all the observations from either the mean or the median of the sample, and an efficiency of .90 by using eight order statistics.

5. Estimation of the correlation coefficient. In the present section we consider the estimation of the correlation coefficient of a normal bivariate population:

$$f(x, y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)}\left(\frac{(x-a)^2}{\sigma_x^2} + \frac{(y-b)^2}{\sigma_y^2} - \frac{2\rho(x-a)(y-b)}{\sigma_x\sigma_y}\right)\right]. \tag{20}$$

The efficient estimate of ρ in a sample $O_n : (x_1', y_1'), (x_2', y_2'), \cdots, (x_n', y_n')$ drawn from the density (20) is

$$r = \frac{\sum(x_i' - \bar{x}')(y_i' - \bar{y})}{[\sum(x_i' - \bar{x})^2 \sum(y_i' - \bar{y})^2]^{\frac{1}{2}}}. \tag{21}$$

There are numerous other techniques in the literature for estimating ρ, among them i) the tetrachoric correlation coefficient which depends on a four-fold table, ii) the adjusted rank correlation coefficient which depends on assigning ranks to the x and y observations. These and other estimates of the correlation coefficient are discussed by Kendall [10].

We shall be concerned with the construction of some estimates of the correlation coefficient which are particularly adapted for use with punched-card equipment. A counting sorter is adequate for the first two cases discussed; in line with our previous development we shall then consider a technique which uses simple addition of the observed values, but does not require sums of squares or products (in the special case where variances of x and y are equal).

5A. Estimation of ρ when means and standard deviations are known. Let us suppose that the means and variances of the variables x and y, distributed according to (20) are given, and consider the problem of estimating the correlation coefficient ρ from a sample of size n. There will be no generality lost by assuming $a = b = 0$, $\sigma_x^2 = \sigma_y^2 = 1$. The technique used will be to construct lines $y = 0$, $x = \pm k$, which cut the xy-plane into six parts. We will form an estimate of ρ based upon the number of observations falling in the four corners. Figure 1 represents the lines laid out in the manner suggested in connection with a scatter diagram of 25 observations; naturally the method is recommended for

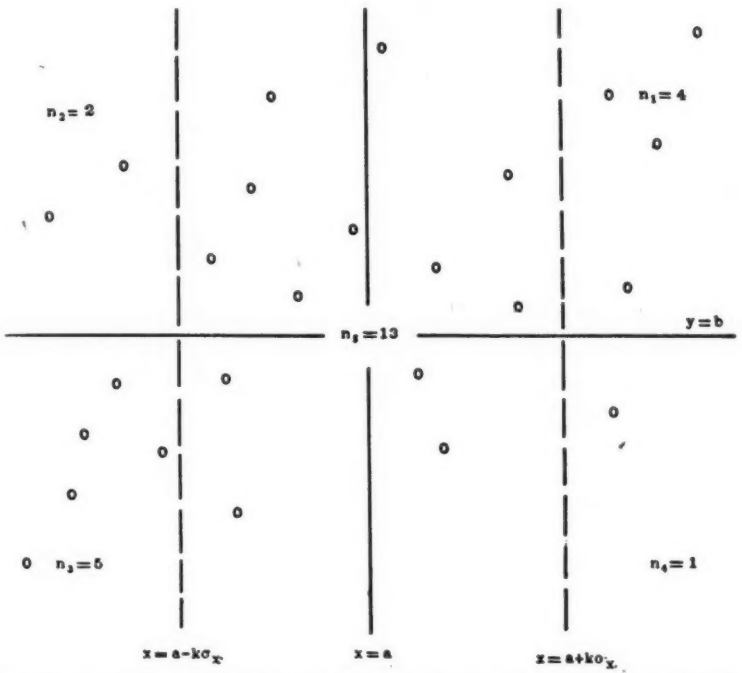

FIG. 1. DIAGRAM OF THE CONSTRUCTION DESCRIBED IN PARAGRAPH 5A WITH A SAMPLE OF 25 OBSERVATIONS SUPERIMPOSED

use only with large samples, the 25 observations are for purposes of illustration only. More specifically after assigning the special values mentioned immediately above to the means and variances in (20), we define

$$p_1 = \int_0^\infty \int_k^\infty f(x, y)\, dx\, dy, \qquad p_3 = \int_{-\infty}^0 \int_{-\infty}^{-k} f(x, y)\, dx\, dy,$$

$$p_2 = \int_0^\infty \int_{-\infty}^{-k} f(x, y)\, dx\, dy, \qquad p_4 = \int_{-\infty}^0 \int_k^\infty f(x, y)\, dx\, dy, \tag{22}$$

$$p_5 = \int_{-\infty}^\infty \int_{-k}^k f(x, y)\, dx\, dy = \int_{-k}^k N(x, 0, 1)\, dx.$$

We denote by n_i the number of observations falling into the region containing probability density p_i. Of course $\sum_{i=1}^{5} n_i = n$. Now we may write the joint probability distribution of the n_i as

$$g(n_1, n_2, n_3, n_4) = \frac{n!}{\prod_1^5 n_i!} \prod_{i=1}^{5} p_i^{n_i}. \tag{23}$$

remembering that $n_5 = n - \sum_1^4 n_i$.

We shall now derive the maximum likelihood estimate of ρ from (23). Taking the logarithm of (23) we have

$$\log g = \log c + \sum_{i=1}^{5} n_i \log p_i, \tag{24}$$

where c is the multinomial coefficient on the right of (23). Differentiating (24) with respect to ρ gives

$$\frac{d(\log g)}{d\rho} = \sum_{i=1}^{4} \frac{n_i \dot{p}_i}{p_i}. \tag{25}$$

where $\dot{p}_i = \frac{dp_i}{d\rho}$; of course $\frac{dp_5}{d\rho} = 0$ because p_5 is functionally independent of ρ. To get $\tilde{\rho}$, the maximum likelihood estimate of ρ, under our restrictions, we must equate the right of (25) to zero and solve for ρ. Before proceeding it will be useful to note the following relations:

$$p_1 = p_3 \; ; \; p_2 = p_4$$

$$\dot{p}_1 = -\dot{p}_4 \; ; \; \dot{p}_2 = -\dot{p}_3 \; ; \; \dot{p}_1 = \dot{p}_3 \; ; \; \dot{p}_2 = \dot{p}_4 \tag{26}$$

$$p_1 + p_4 = \int_k^\infty N(x, 0, 1)dx = \lambda; \qquad p_2 + p_3 = \int_{-\infty}^{-k} N(x, 0, 1)dx = \lambda.$$

If after making appropriate substitutions from (26) we set the right of (25) equal to zero we get

$$\frac{n_1 \dot{p}_1}{p_1} - \frac{n_2 \dot{p}_1}{\lambda - p_1} + \frac{n_3 \dot{p}_1}{p_1} - \frac{n_4 \dot{p}_1}{\lambda - p_1} = 0,$$

and since in general $\dot{p}_1 \neq 0$, the condition is that

$$\frac{n_1 + n_3}{n_2 + n_4} = \frac{p_1}{\lambda - p_1}. \tag{27}$$

Unless all four of the n_i are zero (which is unlikely for reasonable values of λ because n is large), it is possible to find a value of ρ which will make the right side of (27) equal to the ratio formed from the observations on the left, and the value of ρ so determined is the maximum likelihood estimate ρ under the restrictions we have imposed. In practice this equation may be solved by con-

sulting a table of the bivariate normal distribution—see for example K. Pearson [14]. Alternatively [27] may be solved by referring to Figure 3. Truman Kelley [9, 1939] has considered a closely related problem in connection with the validation of test items.

It may be inquired whether it would not be preferable to reduce the present design to a tetrachoric case by using only the cutting lines $x = 0$, $y = 0$. An investigation of the variance of $\tilde{\rho}$ reveals that such is not the case. We proceed to determine the asymptotic variance by means of the usual maximum likelihood technique. Differentiating (25) once more we have

$$\frac{d^2(\log g)}{d\rho^2} = \sum_{i=1}^{4} \frac{n_i(p_i \ddot{p}_i - \dot{p}_1^2)}{p_1^2}, \tag{28}$$

where $\ddot{p}_1 = \dfrac{d^2 p_1}{d\rho^2}$. We note that $E(n_i) = np_i$, therefore

$$E\left(\frac{d^2(\log g)}{d\rho^2}\right) = n\left(\sum_{i=1}^{4} \ddot{p}_i - \sum_{i=1}^{4} \frac{\dot{p}_1^2}{p_i}\right). \tag{29}$$

but since the derivative of a sum is equal to the sum of its derivatives, and $p_1 + p_4 = \lambda$, $p_2 + p_3 = \lambda$, the first sum in the square brackets vanishes. Suitable substitutions from (26) will reduce the second sum so that we get

$$-E\left[\frac{d^2(\log g)}{d\rho^2}\right] = \frac{2n\dot{p}_1^2\lambda}{p_1(\lambda - p_1)}. \tag{30}$$

Therefore asymptotically $\tilde{\rho}$ is normally distributed with variance

$$\sigma^2(\tilde{\rho}) = \frac{p_1(\lambda - p_1)}{2n\lambda\dot{p}_1^2}. \tag{31}$$

In general the optimum value (in the minimum variance sense) of λ which determines the cutting lines $x = \pm k$ will depend on the true value of ρ. To carry out the minimization process in general will require fairly extensive computations, which we feel would be justified. For the present we shall restrict ourselves to minimizing $\sigma^2(\tilde{\rho})$ for the case $\rho = 0$.

We have

$$\dot{p}_1 = \frac{1}{2\pi}\exp\left[-\tfrac{1}{2}k^2\right] = \frac{1}{\sqrt{2\pi}} f(k).$$

when $\rho = 0$, and $p_1 = \frac{1}{2}\lambda$. This gives

$$\sigma^2(\tilde{\rho} \mid \rho = 0) = \frac{\pi\lambda}{4n[f(k)]^2}. \tag{32}$$

We wish to minimize the expression on the right. We recall that a similar expression λ_1/f_1^2 was to be minimized in section 3 when the optimum pair of

observations for estimating the mean of a normal distribution was found. Using the previous results we have $\lambda \doteq .2702$, $k = .6121$; which gives us finally

$$\sigma^2_{\text{opt}}(\tilde{\rho} \mid \rho = 0) \doteq \frac{1.939}{n}. \tag{33}$$

To summarize: if a sample of size n is drawn from a normal bivariate population with known means a_x, a_y and variances σ_x^2 and σ_y^2, but unknown correlation ρ, the maximum likelihood estimate of ρ based on the number of observations falling in the four corners of the plane determined by the lines $x = a_x \pm k\sigma_x$, $y = a_y$ is found by solving for ρ the equation

$$\frac{n_1 + n_3}{n_1 + n_2 + n_3 + n_4} = \frac{p_1}{\lambda}.$$

where n_1 is the number of observations falling in the upper right, n_2 in the upper left, n_3 in the lower left, n_4 in the lower right hand corner, and p_i is the probability density in the region into which the n_i fall, $\lambda = p_1 + p_4$. The variance of this estimate $\tilde{\rho}$ is given by

$$\sigma^2(\tilde{\rho}) = \frac{p_1(\lambda - p_1)}{2n\lambda p_1^2},$$

which is minimized for $\rho = 0$ by setting $k = .6121$, $\lambda = .2702$, giving

$$\sigma^2_{\text{opt}}(\tilde{\rho} \mid \rho = 0) \doteq \frac{1.939}{n}.$$

On the other hand if the usual tetrachoric estimate is used with $x = 0$, $y = 0$ as the cutting lines we get $\sigma_T^2(\tilde{\rho} \mid \rho = 0) = \pi^2/4n$. The relative efficiency of the tetrachoric compared with the optimum statistic is therefore .787. The variance of the efficient estimate r given in (25) when $\rho = 0$ is $1/n$. Consequently the efficiency of our estimate $\tilde{\rho}$ compared to that of r is about .515 for the special case $\rho = 0$ under consideration. This means about twice as large a sample is required to get the same precision with $\tilde{\rho}$ as with r. Doubling the sample and using the cruder statistic $\tilde{\rho}$ may often be an economical procedure.

It may be surmised that a still better estimate of ρ could be constructed by employing four cutting lines, say $x = \pm k$, $y = \pm k$. The simplifications which we used to obtain the estimate $\tilde{\rho}$ no longer hold when we use this new construction. However, it is still possible to compute the minimum variance of the new estimate which we will call $\tilde{\rho}'$, for the special case $\rho = 0$. It again turns out that $k \doteq .6121$ minimizes and we get

$$\sigma^2_{\text{opt}}(\tilde{\rho}' \mid \rho = 0) \doteq \frac{1.52}{n}, \tag{34}$$

which makes the efficiency of $\tilde{\rho}'$ (compared with r) about .66 as compared with .515 for $\tilde{\rho}$. This suggests that if some very simple technique can be found for obtaining $\tilde{\rho}'$, $\tilde{\rho}'$ would be worth using. Unfortunately the author has not been able to construct a rapid way of finding $\tilde{\rho}'$.

5B. Estimation of ρ when the parameters are unknown. A more practical situation than the case treated in paragraph 5A, is the case in which all parameters of (20) are unknown. This case will be treated by means of order statistics. We construct an order statistic analogue of the estimate $\tilde{\rho}$ which we will call $\tilde{\rho}^*$. In general the procedure will be as follows: Each of the N observations in the sample has an x coordinate and a y coordinate

i) order the observations with respect to the x coordinate;

ii) discard all observations except the n with the largest x coordinates called the *right* set and the n with the smallest x coordinates called the *left* set, retaining, therefore, $2n$ observations;

iii) order the pooled $2n$ observations with respect to the y coordinate;

iv) break the $2n$ observations into two sets of n observations each; the *upper* set containing the n observations with the greatest y coordinates, and the *lower* set containing the n observations with the smallest y coordinates;

v) reorder the *upper* set of observations with respect to the x coordinate; the n observations will be divided into those whose x coordinates belong to the *right* set and those whose x coordinates belong to the *left* set;

vi) the estimate $\tilde{\rho}^*$ will be obtained by solving the equation

$$\frac{n_1^*}{n - n_1^*} = \frac{p_1^*}{\lambda_1^* - p_1^*}. \tag{35}$$

where n_1^* is the number of observations in the *upper* set which are also numbers of the *right* set and p_1^* is $\int_0^\infty \int_{k^*}^\infty f(x, y)dx\,dy$, while $f(x, y)$ is the bivariate normal (20) with $\sigma_x = \sigma_y = 1$, $a = b = 0$, and $\int_{k^*}^\infty N(x, 0, 1)\,dx = \frac{n}{N} = \lambda_1^*$.

Figure 2 represents graphically the construction described above for a scatter diagram composed of 25 observations. Of course the number 25 is only for purposes of illustration, as the method is only proposed for use with large samples.

The procedure of ordering the x's and choosing the right and left sets of observations is analogous to cutting the bivariate distribution by the two lines $x = \pm k$ as described in paragraph 5A, indeed $x = x_{n+1}$ and $x = x_{N-n}$ are the corresponding lines, but they vary from sample to sample. To continue the analogy, ordering the remaining observations with respect to y and dividing them into *upper* and *lower* sets of equal size is like cutting the plane with the line $y = 0$. Finally formula (35) is analogous to formula (27). Another similar change is that where formerly we had among relations (26) the equalities $p_1 =$

p_3, $p_2 = p_4$, we now have the corresponding relations amongst the number of observations in the four corners of the plane, namely $n_1^* = n_3^*$, $n_2^* = n_4^*$ which

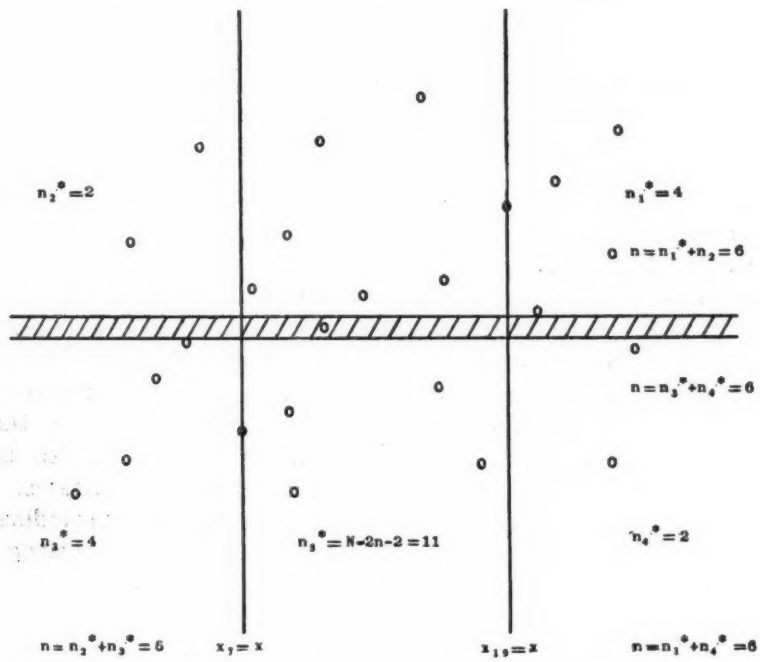

FIG. 2. DIAGRAM OF THE CONSTRUCTION DESCRIBED IN PARAGRAPH 5B ON THE BASIS OF 25 OBSERVATIONS $n = 6$*

can readily be seen by inspection of the fourfold table we have constructed below (omitting all reference to $N - 2n$ pairs of observations we have discarded).

	Left set	Right set	Totals
Upper set	n_2^*	n_1^*	n
Lower set	n_3^*	n_4^*	n
Totals	n	n	$2n$

We have dwelt at length upon the analogy between the two constructions because one of the principal difficulties in working with order statistics is to design a mathematically workable model. The author has found it fruitful when constructing systematic statistics to study a workable analogy which does not involve the order statistics directly, and then to build upon correspondences such as those described.

Some may not wish to read further in this paragraph when they are informed that asymptotically the variance of $\tilde{\rho}^*$ is essentially the same as that of $\tilde{\rho}$. They should proceed to page 404. For the others we proceed to the demonstration.

Suppose we draw a sample of N pairs of observations (x_i', y_i') from the bivariate normal (20). If we discard from these all pairs except those with the n largest x_i and the n smallest x_i, we are left with the right set and the left set. We shall need the joint distribution of x_{n+1} and x_{N-n}

$$J(x_{n+1}, x_{N-n}) = \frac{N!}{(n!)^2(N-2n-2)!}\left(\int_{-\infty}^{x_{n+1}} g(x)\,dx\right)^n \left(\int_{x_{n+1}}^{x_{N-n}} g(x)\,dx\right)^{N-2n-2}\left(\int_{x_{N-n}}^{\infty} g(x)\,dx\right)^n g(x_{n+1}))g(x_{N-n}). \tag{36}$$

where $g(x)$ is the marginal distribution of x obtained from (20), $N(x, a, \sigma_x^2)$. We assume x_{n+1}, x_{N-n} satisfy Condition 1. Considering x_{n+1}, x_{N-n} as fixed and given for the moment we wish to look at the distribution of the y coordinates. We may consider the y coordinates of the observations in the right set as drawn from the distribution of y

$$\varphi'(y) = \frac{\int_{x_{N-n}}^{\infty} f(x, y)\,dx}{\int_{-\infty}^{\infty}\int_{x_{N-n}}^{\infty} f(x, y)\,dx\,dy} = \frac{\int_{x_{N-n}}^{\infty} f(x, y)\,dx}{\int_{x_{N-n}}^{\infty} g(x)\,dx}.$$

Similarly the y coordinates of the observations belonging to the left set may be considered as independently drawn from

$$\psi'(y) = \frac{\int_{-\infty}^{x_{n+1}} f(x, y)\,dx}{\int_{-\infty}^{\infty}\int_{-\infty}^{x_{n+1}} f(x, y)\,dx\,dy} = \frac{\int_{-\infty}^{x_{n+1}} f(x, y)\,dx}{\int_{-\infty}^{x_{n+1}} g(x)\,dx}.$$

To prevent confusion, in considering the y order statistics of the two sets, we shall designate those of the observations which are members of the right set by $u_1, u_2, \cdots, u_n$; while those observations belonging to the left set will have their ordered y coordinates designated $v_1, v_2, \cdots, v_n$. Of course the u's and v's separately satisfy an order relation like that given in (1).

The first question we answer is: given x_{n+1}, x_{N-n}, what is the probability that when we collate the u's and v's and split the observations into the upper set and lower set (see iv). there will be exactly c observations in the lower set whose y coordinates are designated by u's? In other words what is the probability that exactly c members of the lower set belong to the right set? An example for small values of n may clarify the problem. Suppose $n = 4$, and we observe $u_1 < v_1 < v_2 < v_3 < u_2 < u_3 < v_4 < u_4$; the y coordinates of the lower set of observations are u_1, v_1, v_2, v_3, and only the observation with u_1 for its y coordinate belongs to the right set, so for this case $c = 1$. To return

to our general problem, the probability that there are exactly c observations which are members of both the right set and the lower set is

$$P(c \mid x_{n+1}, x_{N-n}) = 1 - p(v_{n-c} > u_{c+1}) - p(u_c > v_{n-c+1}), \tag{37}$$

where $p(w > z)$ is the probability that w is greater than z. Now writing $\varphi(z) = \int_{-\infty}^{z} \varphi'(t)dt$; $\psi(z) = \int_{-\infty}^{z} \psi'(t)dt$ we may rewrite (37) as

$$\begin{aligned} &P(c \mid x_{n+1}, x_{N-n}) \\ &= 1 - \frac{n!}{c!(n-c-1)!} \int_{u_{c+1}}^{\infty} [\psi(v_{n-c})]^{n-c-1}[1 - \psi(v_{n-c})]^{c} \psi'(v_{r-c})\, dv_{n-c} \\ &- \frac{n!}{(n-c)!(c-1)!} \int_{-\infty}^{u_c} [\psi(v_{n-c+1})]^{n-c}[1 - \psi(n_{n-c+1})]^{c-1} \psi'(v_{n-c+1})\, dv_{n-c+1}. \end{aligned} \tag{38}$$

After integrating the first integral of (38) by parts and simplifying we can rewrite (38) as

$$\begin{aligned} P(c \mid x_{n+1}, x_{N-n}) &= \frac{n!}{c!(n-c)!} [\psi(u_{c+1})]^{n-c}[1 - \psi(u_{c+1})]^{c} \\ &+ \frac{n!}{(n-c)!(c-1)!} \int_{\psi(u_c)}^{\psi(u_{c+1})} \alpha^{n-c}(1-\alpha)^{c-1}\, d\alpha. \end{aligned} \tag{39}$$

We approximate the integral term of (39) by

$$\frac{n!}{(n-c)!(c-1)!} [\psi(u_{c+1}) - \psi(u_c)][\psi(u_{c+1})]^{n-c}[1 - \psi(u_{c+1})]^{c-1}$$

which leads us to the approximation

$$\begin{aligned} &P(c \mid x_{n+1}, x_{N-n}) \\ &= \frac{n!}{(n-c)!c!} [\psi(u_{c+1})]^{n-c}[1 - \psi(u_{c+1})]^{c-1}[1 + (c-1)\psi(u_{c+1}) - c\psi(u_c)]. \end{aligned} \tag{40}$$

The joint distribution of u_c, u_{c+1} is given by

$$\begin{aligned} &Q(u_c, u_{c+1} \mid x_{N-n}) \\ &= \frac{n!}{(c-1)!(n-c-1)} \varphi(u_c)^{c-1}(1 - \varphi(u_{c+1}))^{n-c-1} \varphi'(u_c)\varphi'(u_{c+1}). \end{aligned} \tag{41}$$

Next we multiply P as given by (40) by Q from (41) and integrate out u_c. This gives us except for terms of $O\left(\frac{1}{n}\right)$ and higher

$$\begin{aligned} &\frac{n!n!}{c!(n-c-1)!c!(n-c)!} [\varphi(u_{c+1})]^{c} \\ &\cdot [1 - \varphi(u_{c+1})]^{n-c-1}[\psi(u_{c+1})]^{n-c}[1 - \psi(u_{c+1})]^{c} \varphi'(u_{c+1}). \end{aligned} \tag{42}$$

When expression (42) is multiplied by (36), we finally get the approximate joint distribution of c, x_{n+1}, x_{N-n}, u_{c+1}.

Before proceeding further we let

$$\varphi(u_{c+1}) = \frac{\int_{-\infty}^{u_{c+1}} \int_{x_{N-n}}^{\infty} f(x, y)\, dx\, dy}{1 - \lambda_2^*} = \frac{p_4^*}{1 - \lambda_2^*},$$

$$\psi(u_{c+1}) = \frac{\int_{-\infty}^{u_{c+1}} \int_{-\infty}^{x_{n+1}} f(x, y)\, dx\, dy}{\lambda_1^*} = \frac{p_3^*}{\lambda_1^*}, \tag{43}$$

where $\lambda_1^* = \int_{-\infty}^{x_{n+1}} g(x)dx$, $\lambda_2^* = \int_{-\infty}^{x_{N-n}} g(x)dx$. If we also let $p_1^* = 1 - \lambda_2^* - p_4^*$, $p_2^* = \lambda_1^* - p_2^*$ we can write

$$R(c,\ x_{n+1},\ x_{N-n},\ u_{c+1}) = K(\lambda_2^* - \lambda_1^*)^{N-2n-2} p_1^{*n-c-1} p_2^{*c} p_3^{*n-c} p_4^{*c} p_4^{*\prime} \lambda_1^{*\prime} \lambda_2^{*\prime}, \tag{44}$$

where the primes indicate derivatives of p_4^*, λ_1^*, λ_2^* with respect to the appropriate suppressed variables, u_{c+1}, x_{n+1}, x_{N-n}, respectively.

We now proceed to the maximum likelihood estimate of ρ. We take the logarithm of (44) and then take partial derivatives with respect to a the mean of x, b the mean of y, and ρ the correlation coefficient. After equating these partial derivatives to zero we have the following three maximum likelihood equations which must be solved simultaneously to obtain the estimates $\hat{a}^*$, $\hat{b}^*$, and $\hat{\rho}^*$:

$$\frac{1}{N}\left[\frac{N - 2n - 2}{\lambda_2^* - \lambda_1^*}\frac{\partial(\lambda_2^* - \lambda_1^*)}{\partial a} + \frac{n - c - 1}{p_1^*}\frac{\partial p_1^*}{\partial a} + \frac{c}{p_2^*}\frac{\partial p_2^*}{\partial a} + \frac{n - c}{p_3^*}\frac{\partial p_3^*}{\partial a} + \frac{c}{p_4^*}\frac{\partial p_4^*}{\partial a}\right] = 0, \tag{45}$$

$$\frac{1}{N}\left[\frac{n - c - 1}{p_1^*}\frac{\partial p_1^*}{\partial b} + \frac{c}{p_2^*}\frac{\partial p_2^*}{\partial b} + \frac{n - c}{p_3^*}\frac{\partial p_3^*}{\partial b} + \frac{c}{p_4^*}\frac{\partial p_4^*}{\partial b}\right] = 0, \tag{46}$$

$$\frac{1}{N}\left[\frac{n - c - 1}{p_1^*}\frac{\partial p_1^*}{\partial \rho} + \frac{c}{p_2^*}\frac{\partial p_2^*}{\partial \rho} + \frac{n - c}{p_3^*}\frac{\partial p_3^*}{\partial \rho} + \frac{c}{p_4^*}\frac{\partial p_4^*}{\partial \rho}\right] = 0. \tag{47}$$

where terms $O\left(\frac{1}{N}\right)$ have been neglected. Equations (45) and (46) are satisfied, again except for terms $O\left(\frac{1}{N}\right)$, when $\hat{a}^* = \frac{1}{2}(x_{n+1} + x_{N-n})$, $\hat{b}^* = u_{c+1}$. Using this information we examine (47) and find it satisfied when

$$\frac{n - c}{c} = \frac{p_1^*}{\lambda_1^* - p_1^*}, \tag{48}$$

which is directly analogous to equation (27), and is the form promised in (35), if $n_1^* = n - c$. The estimate $\hat{\rho}^*$ is obtained by solving (48) for ρ, where $p_1^* =$

$\int_0^\infty \int_{k^*}^\infty f(x, y)\, dx\, dy$, and $f(x, y)$ is given by (20) with variances equal unity and means equal zero, and $\int_{k^*}^\infty g(x)\, dx = \lambda_1^* = 1 - \lambda_2^* = n/N$.

We shall not go through the derivation of $\sigma^2(\tilde{\rho}^*)$ here. The usual maximum likelihood technique may be used. It turns out that the covariances between $\hat{a}^*$ and $\tilde{\rho}^*$ and between $\hat{b}^*$ and $\tilde{\rho}^*$ are $O\left(\frac{1}{N^2}\right)$. Neglecting such terms we find that the variance is

$$\sigma^2(\tilde{\rho}^*) = \frac{p_1^*(\lambda_1^* - p_1^*)}{2N\lambda_1^* \dot{p}_1^{*2}}. \tag{49}$$

To summarize: if a sample of size N is drawn from a normal bivariate population with unknown parameters, the maximum likelihood estimate of ρ based on the $2n$ observations composed of those observations with the n largest x coordinates and the n smallest x coordinates, may be obtained by solving for ρ the equation

$$\frac{n - c}{n} = \frac{p_1^*}{\lambda^*},$$

where $\frac{1}{2} > \lambda^* = n/N > 0$, $p_1^* = \int_0^\infty \int_{k^*}^\infty f(x, y \mid \sigma_x = 1, a_x = a_y = 0)\, dx\, dy$, $\int_{k^*}^\infty N(x, 0, 1)\, dx = \lambda^*$, *and $n - c$ is the number of the $2n$ observations with largest y coordinates, which also have largest x coordinates. The variance of this estimate $\tilde{\rho}^*$ is given by*

$$\sigma^2(\tilde{\rho}^*) = \frac{p_1^*(\lambda_1^* - p_1^*)}{2N\lambda^* \dot{p}_1^{*2}},$$

and for $\rho = 0$ the variance is minimized by choosing $\lambda^* = .2702$, that is by choosing that 27 per cent of the observations with largest x coordinates, and that 27 per cent with smallest x coordinates, and for this value of λ^*

$$\sigma^2_{\text{opt}}(\tilde{\rho}^* \mid \rho = 0) \doteq \frac{1.939}{N}.$$

Equation (49) is of course exactly analogous to the expression given in (31) for the case of known means and variances. Therefore if the variance minimization problem is solved in general for the case of paragraph 5A, the large sample solution of the problem for unknown means and variances will also be solved.

Figure 3 may be used to obtain the estimates $\tilde{\rho}$ or $\tilde{\rho}^*$ in case the methods of paragraphs 5A or 5B are used. Essentially the figure solves equations (27) and (48). The procedure for the problem of paragraph 5A is

i) when $n_1 + n_3 > n_2 + n_4$ evaluate the ratio $\dfrac{n_1 + n_3}{n_1 + n_2 + n_3 + n_4} = x_0$ and

find the intersection of the line $x = x_0$ with the curve for the particular λ being used;

ii) through the point of intersection of the vertical line $x_0 = x$ and the λ curve draw a horizontal line;

iii) the value of $\tilde{\rho}$ is indicated on the vertical axis at the point of intersection of the horizontal line and the vertical axis;

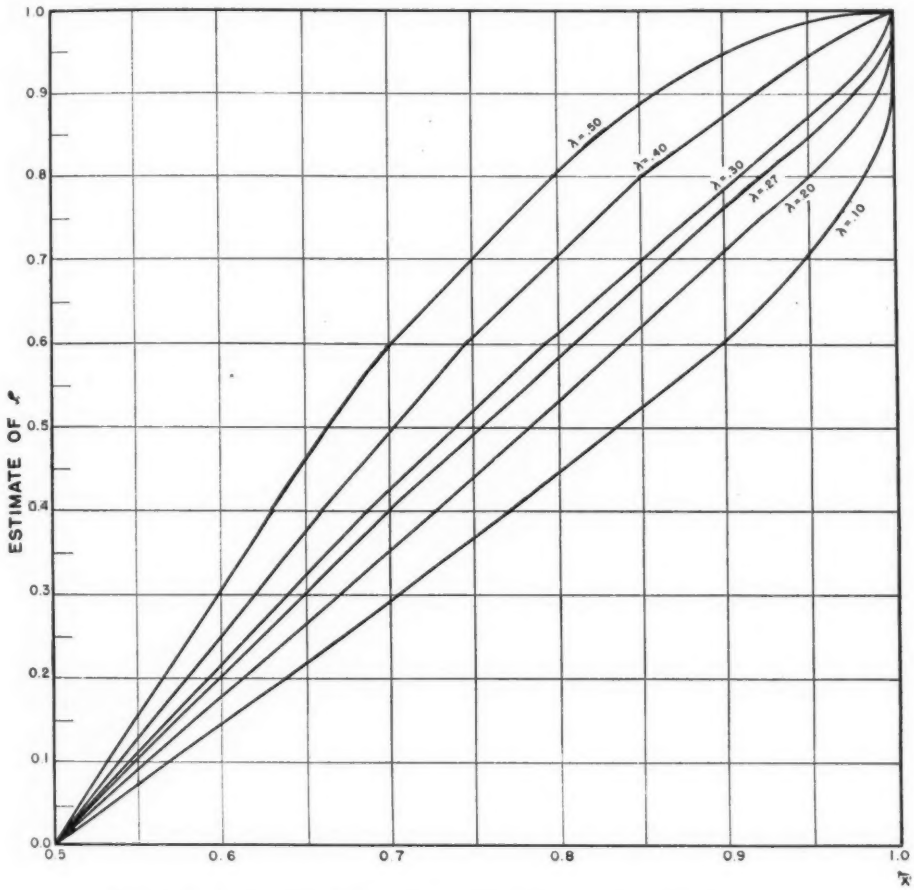

FIG. 3. CURVES FOR ESTIMATING THE CORRELATION COEFFICIENT ρ

iv) when $n_1 + n_3 < n_2 + n_4$ use the ratio $x_0 = \dfrac{n_2 + n_4}{n_1 + n_2 + n_3 + n_4}$ and follow the same procedure, $\tilde{\rho}$ will be the negative of the number appearing on the vertical axis.

Example. Suppose a sample of 1000 is drawn from a normal bivariate population for which the mean of x is a, and the mean of y is b, and the variance of x is σ_x^2, all three parameters known (it is not necessary to know σ_y^2). The xy

plane is cut by the three lines $x = a \pm k\sigma_x$, $y = a$, where, say, $k = .612$, so that $\lambda = .27$. Suppose we find the observations are distributed as follows:

in the upper right-hand corner: $160 = n_1$

in the lower left-hand corner: $170 = n_3$

in the upper left-hand corner: $110 = n_2$

in the lower right-hand corner: $110 = n_4$.

To estimate $\tilde{\rho}$ we set up $x_0 = (n_1 + n_3)/(n_1 + n_2 + n_3 + n_4) = 330/550 = .6$. Referring to Figure 3 we find that the estimate of ρ, $\tilde{\rho} = .20$.

In using Figure 3 for this case it is useful to know that for

$\lambda = .50$	$k = 0.000$	$\lambda = .27$	$k = 0.612$
$\lambda = .40$	$k = 0.253$	$\lambda = .20$	$k = 0.841$
$\lambda = .30$	$k = 0.524$	$\lambda = .10$	$k = 1.282$

If the means and variances of the variables are unknown, we may use the method of paragraph 5B:

i) when $n - c > c$ evaluate the ratio $(n - c)/n = x_0$, and find the intersection of the line $x = x_0$ with the curve for the particular λ_1 being used;

ii) through the point of intersection of the vertical line $x_0 = x$ and the λ_1 curve draw a horizontal line;

iii) the value of $\tilde{\rho}^*$ is indicated on the vertical axis at the point of intersection of the horizontal line and the vertical axis;

iv) when $n - c < c$, use the ratio $c/n = x_0$ and follow the same procedure, $\tilde{\rho}^*$ will be the negative of the number appearing on the vertical axis.

Example: Suppose a sample of 1000 is drawn from a normal bivariate population with all parameters unknown. Suppose we set $n = 200$, and follow the procedure given in paragraph 5B of this section, and suppose we find the observations are distributed as follows:

in the upper right-hand corner: $50 = n - c$

then of course

in the lower left-hand corner: $50 = n - c$

in the upper left-hand corner: $150 = c$

in the lower right-hand corner: $150 = c$

The estimate this time is clearly negative, so we set $x_0 = c/n = 150/200 = .75$. Referring to Figure 3 we find using the curve corresponding to $\lambda = .20$ that the estimate of ρ, $\tilde{\rho} = -.44$.

5C. The use of averages for estimating ρ when the variance ratio is known. Nair and Shrivastava [12, 1942] have considered the use of means for estimating

regression coefficients when one observation is taken at each of n equally spaced fixed variates, x_i $(i = 1, 2, \cdots, n)$, and y is normally distributed. Their procedure was essentially to consider the ordered fixed variates, and to discard a group of observations in the interior, much as we discarded the set of observations whose x coordinates were $x_{n+1}, x_{n+2}, \cdots, x_{N-n}$ in paragraph 5B. The resulting estimates depended essentially on the averages of the y's on the right and left sets of observations, and on the averages of the fixed x's in the two sets.

In an unpublished manuscript George Brown has considered a problem even more closely related to the one considered in paragraph 5A. Suppose x and y normally distributed according to (20) with equal variances σ^2, and means equal to zero. (The ratio of variances must be known, equality is unnecessary.) Retain only those observations for which $| x_i | \geq k\sigma$, and from them form the statistic

$$\rho_B = \frac{\bar{y}_+ - \bar{y}_-}{\bar{x}_+ - \bar{x}_-}, \tag{50}$$

where $\bar{y}_+$ and $\bar{x}_+$ are the average of the n_1 x's and y's for which $x_i > k\sigma$ and $\bar{y}_-$ and $\bar{x}_-$ are similarly defined for the n_2 observations for which $x_i < -k\sigma$. Then ρ_B is an unbiased estimate of ρ. Regarding the x's as fixed variates it turns out that

$$\sigma^2(\rho_B) = \frac{(1 - \rho^2)\sigma^2}{(\bar{x}_+ - \bar{x}_-)^2}\left(\frac{1}{n_1} + \frac{1}{n_2}\right). \tag{51}$$

If we approximate by substituting expected values for observed values (55) turns out to be $(1 - \rho^2)\sigma^2\lambda/2N[g(k)]^2$, where $\lambda = \int_{-\infty}^{-k} g(x)\,dx$, $g(x) = N(x, 0, 1)$. The value of k which minimizes this expression is our old friend $k = .6121$, which gives $\lambda = .2702$. Therefore for $\rho = 0$ and large samples, the minimum variance is approximately $1.23\,\sigma^2/N$, for an efficiency of about .81. The relative efficiency of the methods of paragraphs 5A and 5B are .635 compared with the present technique.

We presume that the analogous order statistics construction would produce much the same result. Our interest in the present technique is to supply an approximate answer to the question of what is to be gained by going from the counting technique proposed in paragraph 5B to the next level of computational difficulty—addition.

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THE NON-CENTRAL WISHART DISTRIBUTION AND CERTAIN PROBLEMS OF MULTIVARIATE STATISTICS[1]

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1. Summary. The non-central Wishart distribution is the joint distribution of the sums of squares and cross-products of the deviations from the sample means when the observations arise from a set of normal multivariate populations with constant covariance matrix but expected values that vary from observation to observation. The characteristic function for this distribution is obtained from the distribution of the observations (Theorem 1). By using the characteristic functions it is shown that the convolution of several non-central Wishart distributions is another non-central Wishart distribution (Theorem 2). A simple integral representation of the distribution in the general case is given (Theorem 3). The integrand is a function of the roots of a determinantal equation involving the matrix of sums of squares and cross-products of deviations of observations and the matrix of sums of squares and cross-products of deviations of corresponding expected values.

The knowledge of the non-central Wishart distribution is applied to two general problems of multivariate normal statistics. The moments of the generalized variance, which is the determinant of sums of squares and cross-products multiplied by a constant, are given for the cases of the expected values of the variates lying on a line (Theorem 4) and lying on a plane (Theorem 5). The likelihood ratio criterion for testing linear hypotheses can be expressed as the ratio of two determinants or as a symmetric function of the roots of a determinantal equation. In either case there is involved a matrix having a Wishart distribution and another matrix independently distributed such that the sum of these two matrices has a non-central Wishart distribution. When the null hypothesis is not true the moments of this criterion are given in the non-central planar case (Theorem 6).

2. Introduction. The well-known Wishart distribution is the distribution of the sums of squares and cross-products of deviations from the sample means of observations from a multivariate normal distribution. If the expected values of the variates change from observation to observation (with the covariance matrix constant), the distribution of sums of squares and cross-products is the *non-central* Wishart distribution. This distribution has been given explicitly [1] for the simple cases of the non-central problem. If we think of the expected values of each observation as defining a point in a space of dimensionality equal to the number of variates, we can say that the cases handled are those in which the points corresponding to a sample lie on a line or

[1] Part of a thesis submitted to the Mathematics Department of Princeton University in partial fulfillment of the requirements for the degree of Doctor of Philosophy, June, 1945.

a plane. Although the explicit formulas for the distribution of higher rank are extremely complicated and have not been derived, the characteristic function is relatively simple. The distribution in general can be given in terms of a simple multiple integral.

The Wishart distribution is the basis of much of the sampling theory associated with the multivariate normal distribution. It plays a role similar to that of the χ^2-distribution in univariate normal theory. It can be used in deriving the distributions of the generalized T^2 and of the multiple correlation coefficient when all variates have a normal distribution; it is used in deriving the moments of the likelihood ratio criterion for testing the general linear hypothesis (including the test of the means of several populations being equal) as well as deriving the moments of other such criteria[2]). For the problems of the T^2 and the test of the linear hypothesis and many other problems, the non-central Wishart distribution must be substituted for the central Wishart distribution when the null hypothesis is not true. That is, the non-central distribution can be the basis of obtaining the power function for many tests in multivariate normal statistics. As an example of the application of the non-central Wishart distribution to these problems, in this paper we obtain the moments of the generalized variance and the moments of the criterion for linear hypotheses when the population means lie on a line or a plane. Applications to other problems such as testing collinearity, comparing scales of measurement, and multiple regression in time series analysis will be published in a later paper [3]. Another problem to which this non-central theory can be applied is a method of estimating the parameters of a single equation of a complete system of linear stochastic difference equations (developed by T. W. Anderson, M. A. Girshick and H. Rubin).

In [1] it was shown that one can make linear transformations on the observations which simplify the derivation of the non-central Wishart distribution in the linear and planar cases. Consider a set of N multivariate normal populations, each of p variates. Let the i-th ($i = 1, 2, \cdots, p$) variate of the α-th ($\alpha = 1, 2, \cdots, N$) population be $x_{i\alpha}$; let the mean of the variate be

$$E(x_{i\alpha}) = \mu_{i\alpha} \qquad (i = 1, 2, \cdots, p; \alpha = 1, 2, \cdots, N). \tag{1}$$

Let the covariance matrix (of rank p) common to all N populations be

$$\| E(x_{i\alpha} - \mu_{i\alpha})(x_{j\alpha} - \mu_{j\alpha}) \| = \| \sigma_{ij} \| \qquad (\alpha = 1, 2, \cdots, N).$$

The probability element of the $x_{i\alpha}$ can be written as

$$| \sigma^{ij} |^{\frac{1}{2}N} (2\pi)^{-\frac{1}{2}pN} \exp\left[-\tfrac{1}{2} \sum_{i,j,\alpha} \sigma^{ij} (x_{i\alpha} - \mu_{i\alpha})(x_{j\alpha} - \mu_{j\alpha}) \right] \prod_{i,\alpha} dx_{i\alpha}, \tag{2}$$

where

$$\| \sigma^{ij} \| = \| \sigma_{ij} \|^{-1}.$$

[2] See Wilks [2] for example.

The sum of squares and cross-products of deviations from the means in a sample $\{x_{i\alpha}\}$ are

$$a_{ij} = \sum_{\alpha=1}^{N} (x_{i\alpha} - \bar{x}_i)(x_{j\alpha} - \bar{x}_j), \tag{3}$$

where

$$\bar{x}_i = \frac{1}{N} \sum_{\alpha=1}^{N} x_{i\alpha}.$$

The dimensionality, say t, of the space spanned by $|| \mu_{i\alpha} ||$ is equal to the rank of

$$|| \tau_{ij} || = || \sum_{\alpha=1}^{N} (\mu_{i\alpha} - \bar{\mu}_i)(\mu_{j\alpha} - \bar{\mu}_j) ||, \tag{4}$$

where

$$\bar{\mu}_{i\alpha} = \frac{1}{N} \sum_{\alpha=1}^{N} \mu_{i\alpha}.$$

As a result of a linear transformation it was demonstrated that the distribution of a_{ij} is the same as that of $\sum_{\alpha=1}^{N-1} x'_{i\alpha} x'_{j\alpha}$ where the $x_{i\alpha}$ have a normal multivariate distribution with covariance matrix $|| \sigma_{ij} ||$ and expected values

$$E(x'_{i\alpha}) = \mu'_{i\alpha}$$
$$(i = 1, 2, \cdots, p; \alpha = 1, 2, \cdots, N-1),$$

such that

$$|| \tau_{ij} || = || \sum_{\alpha=1}^{N-1} \mu'_{j\alpha} ||.$$

The joint distribution of a_{ij} is given for three cases:

(i) *Case* $t = 0$:

$$W(a_{ij}, \sigma_{ij}, \tau_{ij}; p, N-1, 0) = K_0 \, | \sigma^{ij} |^{\frac{1}{2}(N-1)} \, | a_{ij} |^{\frac{1}{2}(N-p-2)} \exp[-\tfrac{1}{2} \sum_{i,j} \sigma^{ij} a_{ij}]; \tag{5}$$

(ii) *Case* $t = 1$:

$$W(a_{ij}, \sigma_{ij}, \tau_{ij}; p, N-1, 1) = K_1 \exp[-\tfrac{1}{2} \sum_{i,j} \sigma^{ij} \tau_{ij}] \, | \sigma^{ij} |^{\frac{1}{2}(N-1)} \, | a_{ij} |^{\frac{1}{2}(N-p-2)} \tag{6}$$
$$\times \exp[-\tfrac{1}{2} \sum_{i,j} \sigma^{ij} a_{ij}] [\sum_{i,j} a_{ij} \tau_{ij}]^{-\frac{1}{4}(N-3)} \, \mathrm{I}_{\frac{1}{2}(N-3)}(\sqrt{\sum_{i,j} a_{ij} \tau_{ij}});$$

(iii) *Case* $t = 2$:

$$W(a_{ij}, \sigma_{ij}, \tau_{ij}; p, N-1, 2) = K_2 \exp[-\tfrac{1}{2} \sum_{i,j} \sigma^{ij} \tau_{ij}] \, | \sigma^{ij} |^{\frac{1}{2}(N-1)}$$
$$\times | a_{ij} |^{\frac{1}{2}(N-p-2)} \exp[-\tfrac{1}{2} \sum_{i,j} \sigma^{ij} a_{ij}] \sum_{w=0}^{\infty} \frac{(u_1 u_2)^w}{2^{2w} w! \Gamma(\frac{1}{2}[N-2] + w)} \tag{7}$$
$$\times (u_1 + u_2)^{-\frac{1}{2}(\frac{1}{2}[N-3]+2w)} \, \mathrm{I}_{\frac{1}{2}(N-3)+2w}(\sqrt{u_1 + u_2}),$$

where

$$K_0^{-1} = 2^{\frac{1}{2}p(N-1)}\,\pi^{\frac{1}{4}p(p-1)}\prod_{i=1}^{p}\Gamma(\tfrac{1}{2}[N-i]),$$

$$K_1^{-1} = 2^{\frac{1}{2}p(N-1)-\frac{1}{2}(N-3)}\,\pi^{\frac{1}{4}p(p-1)}\prod_{i=1}^{p-1}\Gamma(\tfrac{1}{2}[N-1-i]),$$

$$K_2^{-1} = 2^{\frac{1}{2}p(N-1)-\frac{1}{2}(N-3)}\,\pi^{\frac{1}{4}p(p-1)}\prod_{i=1}^{p-2}\Gamma(\tfrac{1}{2}[N-2-i]),$$

$I_n(x)$ is the Bessel function of purely imaginary argument, and u_1 and u_2 are the two non-zero roots of

$$|\,\mathbf{T} - \lambda A^{-1}\,| = 0 \tag{8}$$

(here $\mathbf{T} = ||\,\tau_{ij}\,||$ and $A = ||\,a_{ij}\,||$). The number $N - 1$ is the *number of degrees of freedom* and t is the *rank*. The matrix $||\,\sigma_{ij}\,||$ we shall call the *sigma matrix*, and $||\,\tau_{ij}\,||$ we shall call the *means sigma matrix*.

Let $\kappa_1^2, \kappa_2^2, \cdots, \kappa_p^2$ be the real, non-negative roots of the determinantal equation

$$|\,\mathbf{T} - \lambda\Sigma\,| = 0 \tag{9}$$

(where $\Sigma = ||\,\sigma_{ij}\,||$). There is a non-singular $p \times p$ matrix Ψ $(= ||\,\psi_{ij}\,||)$ such that

$$\Psi\Sigma\Psi' = I \tag{10}$$

and

$$\Psi T\Psi' = ||\,\kappa_i^2\delta_{ij}\,|| \tag{11}$$

(where I is the identity; Ψ' is the transpose of Ψ and $\delta_{ij} = 1$ for $i = j$ and 0 for $i \neq j$). Then the quantities

$$b_{ij} = \sum_{h,k=1}^{p}\psi_{ih}\,\psi_{jk}\,a_{hk} \tag{12}$$

have the distribution $W(b_{ij}, \delta_{ij}, \kappa_i^2\delta_{ij}; p, n, t)$ where $n = N - 1$ and $\kappa_i^2 = 0$ for $i = t + 1, t + 2, \cdots, p$). This is the same distribution that would be derived if the b_{ij} were defined by

$$b_{ij} = \sum_{\alpha=1}^{n} y_{i\alpha}\,y_{j\alpha}\,, \tag{13}$$

where the distribution of the $y_{i\alpha}$ is

$$(2\pi)^{-\frac{1}{2}pn}\exp\,[-\tfrac{1}{2}\sum_{i=1}^{p}\sum_{\alpha=1}^{n}(y_{i\alpha} - \kappa_i\,\delta_{i\alpha})]^2\,. \tag{14}$$

This simplified distribution of the observations has been called the canonical form.

3. The characteristic function of the non-central Wishart distribution. We shall find the characteristic function of the a_{ii} and $2a_{ij}$ $(i \neq j)$ as defined in (3). We first obtain the characteristic function of the b_{ii} and $2b_{ij}$ $(i \neq j)$ as defined in (13) and then perform a linear transformation to obtain the characteristic function of the a_{ij}. The characteristic function of the b_{ii} and $2b_{ij}$ $(i \neq j)$ is defined as

$$E\left(\exp\left[i \sum_{i,j=1}^{p} b_{ij}\theta_{ij}\right]\right), \tag{15}$$

where

$$\theta_{ij} = \theta_{ji}$$

and i in the exponent is the imaginary quantity.
We can write (15) as

$$E\left(\exp\left[i \sum_{i,j=1}^{p}\sum_{\alpha=1}^{n} y_{i\alpha}\, y_{j\alpha}\,\theta_{ij}\right]\right)$$
$$= (2\pi)^{-\frac{1}{2}pn} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \exp\left[-\tfrac{1}{2}\sum_{i=1}^{p}\sum_{\alpha=1}^{n}(y_{i\alpha} - \kappa_i\,\delta_{i\alpha})^2 + i\sum_{i,j=1}^{p}\sum_{\alpha=1}^{n} y_{i\alpha}\, y_{j\alpha}\,\theta_{ij}\right]$$
$$\times \prod_{i=1}^{p}\prod_{\alpha=1}^{n} dy_{i\alpha}\,.$$

Let us first integrate the $y_{i\alpha}$ for $i = 1, 2, \cdots, p$ and $\alpha = t+1, t+2, \cdots, n$; that is, make the integration

$$(2\pi)^{-\frac{1}{2}p(n-t)} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty}$$
$$\cdot \exp\left[-\frac{1}{2}\sum_{i=1}^{p}\sum_{\alpha=t+1}^{n} y_{i\alpha}^2 + i\sum_{i,j=1}^{p}\sum_{\alpha=1}^{n} y_{i\alpha}\, y_{j\alpha}\,\theta_{ij}\right] \prod_{i=1}^{p}\prod_{\alpha=t+1}^{n} dy_{i\alpha}\,.$$

This is, however, the characteristic function of a Wishart distribution with $n - t$ degrees of freedom [4], namely

$$|\,\delta_{ij} - 2i\theta_{ij}\,|^{-\frac{1}{2}(n-t)}\,. \tag{16}$$

Now we must make the integration

$$(2\pi)^{-\frac{1}{2}pt} \exp\left[-\frac{1}{2}\sum_{\eta=1}^{t}\kappa_\eta^2\right] \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty}$$
$$\cdot \exp\left[-\frac{1}{2}\sum_{i=1}^{p}\sum_{\eta=1}^{t} y_{i\eta}^2 + i\sum_{i,j=1}^{p}\sum_{\eta=1}^{t} y_{i\eta}\, y_{j\eta}\,\theta_{ij} + \sum_{\eta=1}^{t} y_{\eta\eta}\,\kappa_\eta\right] \times \prod_{i=1}^{p}\prod_{\eta=1}^{t} dy_{i\eta}\,. \tag{17}$$

There is a $p \times p$ matrix $G = \| g_{ij} \|$ such that

$$\sum_{h,k=1}^{p} g_{hi}\, d_{kk}\, g_{kj} = \delta_{ij}\,,$$

where

$$d_{kh} = \delta_{kh} - 2i\theta_{kh}.$$

Let us make the transformation

$$y_{i\eta} = \sum_{h=1}^{p} g_{ih} z_{h\eta} + d^{i\eta} \kappa_\eta,$$

where

$$|| d^{ih} || = || d_{ih} ||^{-1}.$$

Then the exponent of (17) within the integral sign is

$$-\frac{1}{2}\left\{\sum_{i=1}^{p}\sum_{\eta=1}^{t} z_{i\eta}^2 - \sum_{\eta=1}^{t} d^{\eta\eta}\kappa_\eta^2\right\},$$

and the Jacobian of the transformation is

$$| d_{i\eta} |^{-\frac{1}{2}t}.$$

Hence, the integral of (17) is

$$| d^{ij} |^{\frac{1}{2}t} \exp\left[-\frac{1}{2}\left(\sum_{\eta=1}^{t}\kappa_\eta^2 - \sum_{\eta=1}^{t} d^{\eta\eta}\kappa_\eta^2\right)\right]. \tag{18}$$

This result is obviously true if the θ_{ij} are pure imaginary and sufficiently small so that $|| d_{ij} || = || \delta_{ij} - 2i\theta_{ij} ||$ (which is real in this case) is positive definite. For all complex θ_{ij} in a neighborhood of the origin (17) converges because the real part of $|| d_{ij} ||$ is positive definite. Similarly the integral of the derivative with respect to θ_{ij} of the integrand converges for θ_{ij} in this neighborhood. It follows that the (complex) derivative of the characteristic function exists in this neighborhood because the derivative of the integrand is measurable and is absolutely integrable. Therefore, the characteristic function is analytic in a neighborhood of the origin. From this it follows that the characteristic function is analytic in an open set containing the flat space of real θ_{ij}. By analytic continuation, then, (18) is the value of (17) in the open set containing real θ_{ij}. The characteristic function (15) is the product of (16) and (18). Accordingly, we have the result that the characteristic function of the b_{ii} and $2b_{ij}$ $(i \neq j)$ defined by (13) is

$$| d^{ij} |^{\frac{1}{2}n} \exp\left[-\frac{1}{2}\left(\sum_{\eta=1}^{t}\kappa_\eta^2 - \sum_{\eta=1}^{t} d^{\eta\eta}\kappa_\eta^2\right)\right]. \tag{19}$$

It is clear that if $\kappa_\eta = 0$ (for all η), this function reduces to the characteristic function of the Wishart distribution with n degrees of freedom, namely,

$$| \delta_{ij} - 2i\theta_{ij} |^{-\frac{1}{2}n}. \tag{20}$$

It is interesting to note that (19) factors into two parts, one of which is (20) and the other is

$$\exp\left[-\frac{1}{2}\left(\sum_{\eta=1}^{t}\kappa_\eta^2 - \sum_{\eta=1}^{t} d^{\eta\eta}\kappa_\eta^2\right)\right]. \tag{21}$$

The distribution function similarly factors into two parts, one of which is the Wishart distribution, whose characteristic function is (20). Thus the non-central Wishart distribution function is the convolution of a function (central Wishart distribution) and another (the transform of (21) the first of which is a factor of this same non-central Wishart distribution.

In the planar case the characteristic function can be written as

$$\frac{\exp\left[-\frac{1}{2}(\kappa_1^2 - d^{11}\kappa_1^2)\right]}{|\,\delta_{ij} - 2i\theta_{ij}\,|^{\frac{1}{2}n_1}} \cdot \frac{\exp\left[-\frac{1}{2}(\kappa_2^2 - d^{22}\kappa_2^2)\right]}{|\,\delta_{ij} - 2i\theta_{ij}\,|^{\frac{1}{2}n_2}},$$

where $n_1 + n_2 = n$. From this fact it is clear that the distribution for the planar case (if $n \geq 2p + 2$) is a convolution of two distributions each of the linear case.

This deduction can also be made from the distribution (14). Let

$$b'_{ij} = y_{i1}y_{j1} + \sum_{\alpha=3}^{n_1+1} y_{i\alpha}y_{j\alpha},$$

$$b''_{ij} = y_{i2}y_{j2} + \sum_{\alpha=n_1+2}^{n} y_{i\alpha}y_{j\alpha}.$$

Then it is clear that the b'_{ij} has the non-central Wishart distribution with n_1 degrees of freedom and parameter κ_1^2 in the direction of the first coordinate axis, while the b''_{ij} has the non-central Wishart distribution with n_2 degrees of freedom and parameter κ_2^2 in the direction of the second coordinate axis. Since

$$b_{ij} = b'_{ij} + b''_{ij},$$

the distribution of the b_{ij} is a convolution of the distributions of b'_{ij} and b''_{ij}. In general the non-central distribution is the convolution of t distributions of the linear case (provided $n \geq tp + t$).

It is easy to show that if one has two (or more) non-central Wishart distributions of rank 1 with parameters in the same direction, the convolution is again a non-central Wishart distribution with parameter in the same direction. Suppose b'_{ij} and b''_{ij} have non-central Wishart distributions with parameter $\kappa_1'^2$ and $\kappa_1''^2$ in the direction of the first coordinate axes and n_1 and n_2 degrees of freedom respectively. The characteristic functions are

$$|\,d^{ij}\,|^{\frac{1}{2}n_1} \exp\left[-\tfrac{1}{2}(\kappa_1'^2 - d^{11}\kappa_1'^2)\right]$$

and

$$|\,d^{ij}\,|^{\frac{1}{2}n_2} \exp\left[-\tfrac{1}{2}(\kappa_1''^2 - d^{11}\kappa_1''^2)\right].$$

The product is

$$|\,d^{ij}\,|^{\frac{1}{2}n} \exp\left[-\tfrac{1}{2}(\kappa_1^2 - d^{11}\kappa_1^2)\right],$$

where $n = n_1 + n_2$ and $\kappa_1^2 = \kappa_1'^2 + \kappa_1''^2$.

Now let us deduce the characteristic function of the a_{ii} and $2a_{ij}$ $(i \neq j)$. Since by (12) the b's are transforms of the a's we can write $a_{ij} = \sum_{h,k=1}^{p} \psi^{ih}\psi^{jk}b_{hk}$.

Then

$$E\left(\exp\left[i\sum_{i,j=1}^{p} a_{ij}\phi_{ij}\right]\right) = E\left(\exp\left[i\sum_{i,j=1}^{p}\phi_{ij}\psi^{ih}\psi^{jk}b_{hk}\right]\right), \tag{22}$$

where $\phi_{ij} = \phi_{ji}$. If we define

$$\theta_{hk} = \sum_{i,j=1}^{p}\phi_{ij}\psi^{ih}\psi^{jk}, \tag{23}$$

then (22) can be derived by substituting (23) in (19).
Let

$$\Phi = ||\,\phi_{ij}\,||.$$

Then

$$||\,d_{ij}\,|| = D = \Psi'^{-1}(\Sigma^{-1} - 2i\Phi)\Psi^{-1}$$

and

$$D^{-1} = \Psi(\Sigma^{-1} - 2i\Phi)^{-1}\Psi'.$$

The characteristic function of the a's then can be written as

$$\frac{\exp\left[-\frac{1}{2}\{tr(\Psi T\Psi') - tr[\Psi(\Sigma^{-1} - 2i\Phi)^{-1}\Psi'\Psi T\Psi']\}\right]}{\{|\,\Psi'^{-1}\,|\cdot|\,\Sigma^{-1} - 2i\Phi\,|\cdot|\,\Psi^{-1}\,|\}^{\frac{1}{2}n}}$$

using (10) and (11). The denominator is

$$\{|\,\Psi'\,|\cdot|\,\Psi\,|\}^{-\frac{1}{2}n}\,|\,\Sigma^{-1} - 2i\Phi\,|^{\frac{1}{2}n}$$

and the numerator can be written as[3]

$$\exp\left[-\tfrac{1}{2}\{\mathrm{tr}\,(M'\Psi'\Psi M) - \mathrm{tr}\,[M'\Psi'\Psi(\Sigma^{-1} - 2i\Phi)^{-1}\Psi'\Psi M]\}\right]$$

where

$$M = ||\,\mu_{i\alpha} - \mu_i\,||$$

and

$$M'M = T.$$

We may summarize in the following theorem:

THEOREM 1. *Given a_{ij} $(i, j = 1, 2, \cdots, p)$ defined by* (3) *where the $x_{i\alpha}$ $(i = 1, 2, \cdots, p;\ \alpha = 1, 2, \cdots, N)$ are distributed according to* (2), *the characteristic function of a_{ij} and $2a_{ij}$ $(i \neq j)$ is*

$$E\left(\exp\left[i\sum_{i,j=1}^{p} a_{ij}\phi_{ij}\right]\right)$$

$$= \frac{|\,\sigma^{ij}\,|^{\frac{1}{2}(N-1)}\exp\left[-\frac{1}{2}\sum_{i,j=1}^{p}\sum_{\alpha=1}^{N}\sigma^{ij}(\mu_{i\alpha} - \bar{\mu}_i)(\mu_{j\alpha} - \bar{\mu}_j)\right]}{|\,\tilde{\sigma}^{ij}\,|^{\frac{1}{2}(n-1)}} \tag{24}$$

$$\cdot\exp\left[\frac{1}{2}\sum_{\substack{i,j=1\\h,k=1}}^{p}\sum_{\alpha=1}^{N}\sigma^{hi}\tilde{\sigma}_{ij}\sigma^{jh}(\mu_{h\alpha} - \bar{\mu}_h)(\mu_{k\alpha} - \bar{\mu}_k)\right],$$

[3] The result follows from the fact that tr(AB) = tr(BA).

where

$$|| \tilde{\sigma}_{ij} ||^{-1} = || \tilde{\sigma}^{ij} || = || \sigma^{ij} - 2i\phi_{ij} ||$$

and

$$\phi_{ij} = \phi_{ji} .$$

Suppose we have two sets of quantities a'_{ij} and a''_{ij} each set of which is distributed according to a non-central Wishart distribution with sigma matrix $|| \sigma^{ij} ||$, one having n' degrees of freedom (or n''), means sigma matrix τ'_{ij} (or τ''_{ij}) of rank t' (or t''). Consideration of the characteristic functions (24) shows that

$$a_{ij} = a'_{ij} + a''_{ij}$$

has a non-central Wishart distribution with matrix $|| \sigma^{ij} ||$, $n' + n''$ degrees of freedom and a matrix

$$|| \tau_{ij} || = || \tau'_{ij} || + || \tau''_{ij} ||.$$

The rank of the distribution is equal to the rank of $|| \tau_{ij} ||$. This result can also be deduced from the representation of a'_{ij} and a''_{ij} in terms of observations from non-central normal populations. It is a straightforward generalization of the same result for central Wishart distributions.

THEOREM 2. *The convolution of two or more non-central Wishart distributions with identical sigma matrices is a non-central Wishart distribution with means sigma matrix equal to the sum of the means sigma matrices of the components.*

4. An integral representation of the non-central Wishart distribution in the general case. It was shown in [1] that

$$W(b_{ij}, \delta_{ij}, \kappa_i^2 \delta_{ij}; p, n, t) = Ce^{-\frac{1}{2}trB} \int |B - YY'|^{\frac{1}{2}(n-p-t-1)} e^{tr(K'Y)} dY$$

where

$$C^{-1} = e^{\frac{1}{2}tr(K'K)} 2^{\frac{1}{2}pn} \pi^{\frac{1}{4}p(p-1)+\frac{1}{2}pt} \prod_{i=1}^{p} \Gamma(\tfrac{1}{2}[n - t + 1 - i]),$$

$$dB = \prod_{i=1}^{p} \prod_{j=i}^{p} db_{ij},$$

$$dY = \prod_{i=1}^{p} \prod_{\eta=1}^{t} dy_{i\eta},$$

$$B = || b_{ij} ||,$$

$$Y = || y_{i\eta} || \qquad (\eta = 1, 2, \cdots, t),$$

$$K = || \kappa_i \delta_{i\eta} || \qquad (\eta = 1, 2, \cdots, t),$$

and the integration is on Y over the range $|| B - YY' ||$ positive semi-definite. This is equivalent to

$$(25) \qquad Ce^{-\frac{1}{2}trB} \mid B \mid^{\frac{1}{2}(n-p-t-1)} \int \mid I - Y'B^{-1}Y \mid^{\frac{1}{2}(n-p-t-1)} e^{tr(K'Y)}\, dY.$$

The integration is over the range of Y for which $\| I - Y'B^{-1}Y \|$ is positive semi-definite.

There is a p by p matrix $H = \| h_{ij} \|$ such that

$$H'B^{-1}H = I$$

$$H'K = W = \| w_i\delta_{i\eta} \|,$$

where w_i^2 are the roots of

$$\mid \kappa_i^2\delta_{ij} - \lambda b^{ij} \mid = 0,$$

$$\| b^{ij} \| = \| b_{ij} \|^{-1}.$$

Then make the transformation to $Z = \| z_{i\eta} \|$ by

$$Y = HZ.$$

The Jacobian of the transformation is

$$\mid H \mid^t = \mid B \mid^{\frac{1}{2}t}.$$

Then (25) can be written as

$$(26) \qquad Ce^{-\frac{1}{2}trB} \mid B \mid^{\frac{1}{2}(n-p-1)} \int \mid I - Z'Z \mid^{\frac{1}{2}(n-p-t-1)} e^{trW(\prime Z)}\, dZ.$$

Partition

$$Z = \left\| \begin{matrix} Z^1 \\ Z_1 \end{matrix} \right\|$$

such that Z_1 is square $(t \times t)$. Let $I - Z_1'Z_1 = E'E$, (in terms of Z_1), where E is specified uniquely and consider the transformation of variables from Z_2 to V defined by

$$Z_2 = VE.$$

Then (26) can be written as

$$Ce^{-\frac{1}{2}trB} \mid B \mid^{\frac{1}{2}(n-p-1)} dB \int \mid I - Z_1'Z_1 \mid^{\frac{1}{2}(n-2t-1)} e^{tr(W_1'Z_1)}\, dZ_1$$

$$\cdot \int \mid I - V'V \mid^{\frac{1}{2}(n-p-t-1)} dV_1$$

where

$$W_1 = \| w_\eta\delta_{\eta\xi} \| \qquad (\eta, \xi = 1, 2, \cdots, t).$$

The first integration is over the range $(I - Z_1'Z_1)$ positive semi-definite and the second is over $(I - V'V)$ positive semi-definite. The value of the second integral is

$$\int |I - V'V|^{\frac{1}{2}(n-p-t-1)}\, dV = \frac{\pi^{\frac{1}{4}t(p-t)} \prod_{i=1}^{p-t} \Gamma(\frac{1}{2}[n-2t+1-i])}{\prod_{i=1}^{p-t} \Gamma(\frac{1}{2}[n-t+1-i])}.$$

Hence (26) can be written as

$$C_1 e^{-\frac{1}{2}trB} |B|^{\frac{1}{2}(n-p-1)} \int |I - Z_1'Z_1|^{\frac{1}{2}(n-2t-1)} e^{tr(W_1'Z_1)}\, dZ_1 \tag{27}$$

with

$$C_1^{-1} = e^{\frac{1}{2}tr(K'K)} 2^{\frac{1}{2}pn} \pi^{\frac{1}{4}p(p-1)+\frac{1}{2}t^2} \times \prod_{i=1}^{t} \Gamma(\tfrac{1}{2}[n-t+1-i]) \prod_{i=1}^{p-t} \Gamma(\tfrac{1}{2}[n-t+1-i]). \tag{28}$$

The first part of (27) is, except for a constant factor, a central Wishart distribution with n degrees of freedom. The integral of the second part is obviously a symmetric function of the w_i'. In terms of the a_{ij} the w_i^2 are simply the roots of (8). We can sum these results in a theorem.

THEOREM 3. *Given a sample of observations* $\{x_{i\alpha}\}$ $(i = 1, 2, \cdots, p; \alpha = 1, 2, \cdots, N)$ *distributed according to* (2), *the probability density function of the sums of squares and cross products of deviations from the sample means defined by* (3) *is*

$$W(a_{ij}, \sigma_{ij}, \tau_{ij}; p, N-1, t) = C_1 |\sigma^{ij}|^{\frac{1}{2}(N-1)} |a_{ij}|^{\frac{1}{2}(N-p-2)} \cdot \exp\left[-\frac{1}{2}\sum_{i,j} \sigma^{ij}\alpha_{ij}\right] \int \left|\delta_{\eta\xi} - \sum_{i=1}^{t} z_{i\eta} z_{i\xi}\right|^{\frac{1}{2}(N-2t-2)} \cdot \exp \sum_{i=1}^{t} w_i z_{ii} \prod_{\eta,\xi=1}^{t} dz_{\eta\xi}$$

integrated over

$$\left\| \delta_{\eta\xi} - \sum_{i=1}^{t} z_{i\eta} z_{i\xi} \right\|$$

positive semi-definite where C_1 $(n = N-1)$ *and* τ_{ij} *are defined by* (28) *and* (4), *respectively, and where* w_η^2 *are the* t *non-zero roots of* (8).

5. The moments of the generalized variance in the linear and planar non-central cases.

5.1. *The linear case.* The generalized variance, which is the determinant of the variances and covariances,[4] is a measure of the spread of the observations. If one thinks of the N observations of each variate as a vector in N-space with

[4] This definition of Wilks [5] was made in terms of variances and covariances defined by a_{ij}/N (from equation (3)). Since we consider $a_{ij}/(N-1)$ to be the variances and covariances we define $|a_{ij}/(N-1)|$ as the generalized variance.

origin at the sample mean, the generalized variance is proportional to the square of the volume of the p dimensional parallelotope which is defined by these vectors as principal edges. Another geometric interpretation can be given in terms of the p-dimensional variate space. The generalized variance is proportional to the sum of the squared volumes of all possible parallelotopes that can be joined by choosing as the p principal edges p of the N sample vectors (origin at the sample mean).

In this section we consider the moments of the generalized variance when the distributions of the observations are non-central multivariate normal. In terms of the first geometric representation this means that the center of one or more of the vector distributions is different from the others. For convenience we shall assume that the distribution of the observations $\{y_{i\alpha}\}$ is according to (14). This will give as much generality as if we treated observations $\{x_{i\alpha}\}$ having the distribution (2). Moreover, we shall consider the determinant of sums of squares and cross-products instead of the determinant of variances and covariances. It is clear that the determinantant $|b_{ij}|$, defined by (13), is simply a multiple (by $|\Sigma|(N-1)^p$) of $\left|\frac{a_{ij}}{N-1}\right|$, defined by (3).

Let us first consider the linear case, i.e., $\kappa = \kappa_1 \neq 0$ and $\kappa_i = 0$ $(i = 2, \cdots, p)$ in (14). The first of the p vectors is centered on the first coordinate axis, not at the origin. Then the probability density function of the b_{ij} is

$$(29)\qquad \frac{e^{-\frac{1}{2}\kappa^2}\,|b_{ij}|^{\frac{1}{2}(n-p-1)}\exp\left[-\frac{1}{2}\sum_{i=1}^{p} b_{ii}\right]}{2^{\frac{1}{2}pn}\pi^{\frac{1}{4}p(p-1)}\prod_{i=1}^{p-1}\Gamma(\tfrac{1}{2}[n-i])}\sum_{\alpha=0}^{\infty}\frac{(\kappa^2 b_{11})^{\alpha}}{2^{2\alpha}\alpha!\Gamma(\frac{1}{2}n+\alpha)}.$$

We wish to find the moments $E(|b_{ij}|^h)$. Let

$$b_{ij} = s_i s_j r_{ij}.$$

Then s_i^2 is the sum of squares of the i-th variate and $\|r_{ij}\|$ is the matrix of sample correlation coefficients. The Jacobian of this transformation (to s_i^2, r_{ij}) is

$$(s_1^2)^{\frac{1}{2}(p-1)}(s_2^2)^{\frac{1}{2}(p-1)}\cdots(s_p^2)^{\frac{1}{2}(p-1)}.$$

The probability element of the s^2's and r's is

$$(30)\qquad \frac{\exp\left[-\frac{1}{2}\kappa^2-\frac{1}{2}\sum_{i=1}^{p}s_i^2\right]\prod_{i=1}^{p}(s_i^2)^{\frac{1}{2}n-1}}{2^{\frac{1}{2}pn}\pi^{\frac{1}{4}p(p-1)}\prod_{i=1}^{p-1}\Gamma(\frac{1}{2}[n-i])}\,|r_{ij}|^{\frac{1}{2}(n-p-1)}$$

$$\times\sum_{\alpha=0}^{\infty}\frac{(\kappa^2 s_1^2)^{\alpha}}{2^{2\alpha}\alpha!\Gamma(\frac{1}{2}n+\alpha)}\prod_{i=1}^{p}d(s_i^2)\prod_{i=1}^{p}\prod_{j=i+1}^{p}dr_{ij}.$$

It is clear from (30) that the s_i^2 are distributed independently and that the set r_{ij} have a joint distribution independent of the s_i^2's. Hence

$$E(|\, b_{ij} \,|^h) = E(|\, s_i\, s_j\, r_{ij} \,|^h) = \prod_{i=1}^{p} E[(s_i^2)^h] \cdot E(|\, r_{ij} \,|^h).$$

The probability element of s_i^2 $(i = 2, 3, \cdots, p)$ is

$$\frac{e^{-\frac{1}{2}s_i^2}(s_i^2)^{\frac{1}{2}n-1}}{2^{\frac{1}{2}n}\Gamma(\frac{1}{2}n)}\, d(s_i^2),$$

which is simply the χ^2-distribution. The h-th moment of s_i^2 $(i = 2, 3, \cdots, p)$ is

$$E[(s_i^2)^h] = \frac{2^h\Gamma(\frac{1}{2}n + h)}{\Gamma(\frac{1}{2}n)}.$$

The probability element of s_1^2 is

$$\frac{e^{-\frac{1}{2}\kappa^2}(s_1^2)^{\frac{1}{2}n-1}e^{-\frac{1}{2}s_1^2}}{2^{\frac{1}{2}n}} \sum_{\alpha=0}^{\infty} \frac{(\kappa^2 s_1^2)^\alpha}{2^{2\alpha}\alpha!\Gamma(\frac{1}{2}n + \alpha)}\, d(s_1^2). \tag{31}$$

This is the χ'^2-distribution (non-central χ^2-distribution) which was given by Fisher [6]. Applying term-by-term integration (the series converges properly) we get the h-th moment

$$E[(s_1^2)^h] = 2^h e^{-\frac{1}{2}\kappa^2} \sum_{\alpha=0}^{\infty} \frac{(\kappa^2)^\alpha\Gamma(\frac{1}{2}n + h + \alpha)}{2^\alpha\alpha!\Gamma(\frac{1}{2}n + \alpha)}.$$

The probability element of the r_{ij} is the well known distribution of correlation coefficients,

$$\frac{\Gamma^{p-1}(\frac{1}{2}n)\,|\, r_{ij} \,|^{\frac{1}{2}(n-p-1)}}{\pi^{\frac{1}{4}p(p-1)} \prod_{i=1}^{p-1} \Gamma(\frac{1}{2}[n - i])} \prod_{i=1}^{p} \prod_{j=i+1}^{p} dr_{ij}.$$

Since

$$\int \frac{|\, r_{ij} \,|^{\frac{1}{2}(n-p-1)}}{\pi^{\frac{1}{4}p(p-1)}} \prod_{i=1}^{p} \prod_{j=i+1}^{p} dr_{ij} = \frac{\prod_{i=1}^{p-1} \Gamma(\frac{1}{2}[n - i])}{\Gamma^{p-1}(\frac{1}{2}n)},$$

where the integration is over the entire (permissable) range o the r_{ij}, we have as a consequence the h-th moment of the determinant (since n is arbitrary)

$$E(|\, r_{ij} \,|^h) = \frac{\Gamma^{p-1}(\frac{1}{2}n)}{\prod_{i=1}^{p-1} \Gamma(\frac{1}{2}[n - i])} \int \frac{|\, r_{ij} \,|^{\frac{1}{2}(n-p-1)+h}}{\pi^{\frac{1}{4}p(p-1)}} \prod_{i=1}^{p} \prod_{j=i+1}^{p} dr_{ij}$$

$$= \frac{\prod_{i=1}^{p-1} \Gamma(\frac{1}{2}[n - i] + h)\Gamma^{p-1}(\frac{1}{2}n)}{\prod_{i=1}^{p-1} \Gamma(\frac{1}{2}[n - i])\Gamma^{p-1}(\frac{1}{2}n + h)}.$$

Hence, the h-th moment of $|s_i s_j r_{ij}|$ is

$$2^{ph}\frac{\prod_{i=1}^{p-1}\Gamma(\frac{1}{2}[n+2h-i])}{\prod_{i=1}^{p-1}\Gamma(\frac{1}{2}[n-i])}e^{-\frac{1}{2}\kappa^2}\sum_{\alpha=0}^{\infty}\frac{\kappa^{2\alpha}\Gamma(\frac{1}{2}n+h+\alpha)}{2^\alpha\alpha!\Gamma(\frac{1}{2}n+\alpha)}. \tag{32}$$

Let us summarize this in a theorem for the a_{ij}.

THEOREM 4. *If the quantities a_{ij} $(i, j = 1, 2, \cdots, p)$ have the distribution $W(a_{ij}, \sigma_{ij}, \tau_{ij}; p, N-1, 1)$ defined by (6), then the moments of $|a_{ij}|$ are given by*

$$E(|a_{ij}|^h) = |\sigma^{ij}|^h 2^{ph}e^{-\frac{1}{2}\kappa^2}\frac{\prod_{i=1}^{p-1}\Gamma(\frac{1}{2}[N-1-i]+h)}{\prod_{i=1}^{p-1}\Gamma(\frac{1}{2}[N-1-i])}\sum_{\alpha=0}^{\infty}\frac{\kappa^{2\alpha}\Gamma(\frac{1}{2}[N-1]+h+\alpha)}{2^\alpha\alpha!\Gamma(\frac{1}{2}[N-1]+\alpha)},$$

where κ^2 is the non-zero root of (9).

The h-th moment of the generalized variance $|a_{ij}/(N-1)|$ is obtained by dividing the above expression by $(N-1)^{ph}$.

If $\kappa^2 = 0$, expression (32) clearly reduces to the moment given by Wilks [5]

$$2^{ph}\frac{\prod_{i=1}^{p}\Gamma(\frac{1}{2}[n+1-i]+h)}{\prod_{i=1}^{p}\Gamma(\frac{1}{2}[n+1-i])}. \tag{33}$$

The expression (32) gives the moments of the generalized variance when the means of the observations are not fixed, but lie on a line. The distribution of $|b_{ij}|$ is not a simple function even in the central case. However, in any particular case one could find the first few moments of $|b_{ij}|$ and fit a distribution function. It is to be noted that the convergence of the series is nearly as rapid as that for $e^{\frac{1}{2}\kappa^2}$.

5.2. *The planar case.* Next we shall treat the planar case for two dimensions. Suppose that $\kappa_i^2 \neq 0$ $(i = 1, 2)$. The probability density function of b_{11}, b_{12}, and b_{22} is

$$\frac{\exp\left[-\frac{1}{2}(\kappa_1^2+\kappa_2^2)-\frac{1}{2}\sum_{i=1}^{2}b_{ii}\right]}{2^n\sqrt{\pi}}(b_{11}b_{22}-b_{12}^2)^{\frac{1}{2}(n-3)} \tag{34}$$
$$\times\sum_{\alpha,\beta=0}^{\infty}\frac{[\kappa_1^2\kappa_2^2(b_{11}b_{22}-b_{12}^2)]^\alpha(\kappa_1^2 b_{11}+\kappa_2^2 b_{22})^\beta}{2^{4\alpha+2\beta}\alpha!\beta!\Gamma(\frac{1}{2}[n-1]+\alpha)\Gamma(\frac{1}{2}n+2\alpha+\beta)}.$$

Let $b_{11} = s_1^2$, $b_{22} = s_2^2$, and $b_{12} = s_1 s_2 r$. The Jacobian is $s_1 s_2$. The probability element of s_1^2, s_2^2 and r is

$$\frac{e^{-\frac{1}{2}(\kappa_1^2+\kappa_2^2)}}{2^n\sqrt{\pi}}(s_1^2)^{\frac{1}{2}n-1}(s_2^2)^{\frac{1}{2}n-1}(1-r^2)^{\frac{1}{2}(n-3)}e^{-\frac{1}{2}(s_1^2+s_2^2)} \tag{35}$$
$$\times\sum_{\alpha,\beta=0}^{\infty}\frac{(\kappa_1^2\kappa_2^2 s_1^2 s_2^2)^\alpha(1-r^2)^\alpha(\kappa_1^2 s_1^2+\kappa_2^2 s_2^2)^\beta\, d(s_1^2)\, d(s_2^2)\, dr}{2^{4\alpha+2\beta}\alpha!\beta!\Gamma(\frac{1}{2}[n-1]+\alpha)\Gamma(\frac{1}{2}n+2\alpha+\beta)}.$$

We wish to find $E\{[s_1^2 s_2^2(1 - r^2)]^h\}$. Let us first multiply (35) by $(1 - r^2)^h$ and integrate from -1 to $+1$. We then obtain

$$2^{-n} e^{-\frac{1}{2}(\kappa_1^2+\kappa_2^2)} (s_1^2)^{\frac{1}{2}n-1} (s_2^2)^{\frac{1}{2}n-1} e^{-\frac{1}{2}(s_1^2+s_2^2)} \times \sum_{\alpha,\beta=0}^{\infty} \frac{(\kappa_1^2 \kappa_2^2 s_1^2 s_2^2)^\alpha (\kappa_1^2 s_1^2 + \kappa_2 s_2^2)^\beta \Gamma(\frac{1}{2}[n-1]+h+\alpha)\, d(s_1^2)\, d(s_2^2)}{2^{4\alpha+2\beta} \alpha!\beta! \Gamma(\frac{1}{2}[n-1]+\alpha)\Gamma(\frac{1}{2}n+2\alpha+\beta)\Gamma(\frac{1}{2}n+h+\alpha)}.$$

Next we multiply by $(s_1^2)^h (s_2^2)^h$, set $(\kappa_1^2 s_1^2 + \kappa_2^2 s_2^2)^\beta/\beta!$ equal to

$$\sum_{\beta_1+\beta_2=\beta} \frac{(\kappa_1^2 s_1^2)^{\beta_1} (\kappa_2^2 s_2^2)^{\beta_2}}{\beta_1!\beta_2!},$$

and integrate s_1^2 and s_2^2 from 0 to ∞. We obtain

$$E([b_{11}b_{22} - v_{12}^2]^h)$$

$$= 2^{2h} \exp\left[-\tfrac{1}{2}(\kappa_1^2 + \kappa_2^2)\right] \times \sum_{\alpha,\beta_1,\beta_2=0}^{\infty} \frac{(\kappa_1^2)^{\alpha+\beta_1} (\kappa_2^2)^{\alpha+\beta_2} \Gamma(\frac{1}{2}n+h+\alpha+\beta_1)\Gamma(\frac{1}{2}n+h+\alpha+\beta_2)}{2^{2\alpha+\beta_1+\beta_2}\, \alpha!\beta_1!\beta_2!\Gamma(\frac{1}{2}[n-1]+\alpha)\Gamma(\frac{1}{2}n+2a+\beta_1+\beta_2)} \times \frac{\Gamma(\frac{1}{2}[n-1]+h+\alpha)}{\Gamma(\frac{1}{2}n+h+\alpha)}, \tag{36}$$

which is the expected value we are seeking.

Clearly this reduces to a special case of (32) if κ_2^2 is set equal to zero.

Now we consider the planar case in p dimensions. Geometrically we have p vectors in n-space. If the $\{y_{i\alpha}\}$ are distributed according to (14) the mean point (i.e., center of distribution) of the first two vectors is different from the origin, but the mean point of each of the other $p - 2$ vectors is the origin. The vectors are distributed independently. The determinant

$$|b_{ij}| = \left| \sum_{\alpha=1}^{n} y_{i\alpha} y_{j\alpha} \right|$$

is the square of the volume of the parallelopiped which can be expressed as

$$v_1 v_2 \cdots v_p \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{p-1},$$

where v_i is the length of the i-th vector and θ_i is the angle between the $(i+1)$-st vector and the flat space determined by the first i vectors. The distribution of $v_3, \cdots, v_p$ and $\theta_2, \cdots, \theta_{p-1}$ is statistically independent of v_1, v_2, and θ_1; for no matter what the plane of the first two vectors is, the conditional distribution of the other variables is the same. Hence

$$E(|b_{ij}|^h) = E[(v_1 v_2 \sin\theta_1)^{2h}] \cdot E[(v_3 v_4 \cdots v_p \sin\theta_2 \cdots \sin\theta_{p-1})^{2h}].$$

If the y's had simply the distribution

$$\frac{1}{(2\pi)^{\frac{1}{2}pn}} \exp\left[-\tfrac{1}{2}\sum_{i=1}^{p}\sum_{\alpha=1}^{n} y_{i\alpha}^2\right], \tag{37}$$

then the h-th moment of $|\, b_{ij}\,|$ would be (33), and the h-th moment of

$$v_1^2 v_2^2 \sin^2\theta_1 = \begin{vmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{vmatrix}$$

would be

$$2^{2h}\frac{\prod_{i=1}^{2}\Gamma(\frac{1}{2}[n+1-i]+h)}{\prod_{i=1}^{2}\Gamma(\frac{1}{2}[n+1-i])}.$$

Since the distribution of $v_3, v_4, \cdots, v_p$ and $\theta_2, \cdots, \theta_{p-1}$ is the same whether the y's are distributed according to (14) or (37), we have

$$E[(v_3 \cdots v_p \sin\theta_2 \cdots \sin\theta_{p-1})^{2h}] = 2^{h(p-2)}\frac{\prod_{i=3}^{p}\Gamma(\frac{1}{2}[n+2h+1-i])}{\prod_{i=3}^{p}\Gamma(\frac{1}{2}[n+1-i])}. \tag{38}$$

Multiplying (36) by (38) we obtain the h-th moment of $|\, b_{ij}\,|$, namely,

$$E(|\, b_{ij}\,|^h) = 2^{hp}\exp\left[-\tfrac{1}{2}(\kappa_1^2+\kappa_2^2)\right]\frac{\prod_{i=3}^{p}\Gamma(\frac{1}{2}[n+2h+1-i])}{\prod_{i=3}^{p}\Gamma(\frac{1}{2}[n+1-i])}$$

$$\times \sum_{\alpha,\beta_1,\beta_2=0}^{\infty}\frac{(\kappa_1^2)^{\alpha+\beta_1}(\kappa_2^2)^{\alpha+\beta_2}\Gamma(\frac{1}{2}n-h+\alpha+\beta_1)}{2^{2\alpha+\beta_1+\beta_2}\,\alpha!\,\beta_1!\,\beta_2!\,\Gamma(\frac{1}{2}[n-1]+\alpha)}$$

$$\frac{\Gamma(\frac{1}{2}n+h+\alpha+\beta_2)\,\Gamma(\frac{1}{2}[n-1]+h+\alpha)}{\Gamma(\frac{1}{2}n+2\alpha+\beta_1+\beta_2)\Gamma(\frac{1}{2}n+h+\alpha)}.$$

This result may be summarized as follows:

THEOREM 5. *Let the probability density function of the quantities a_{ij} $(i, j = 1, 2, \cdots, p)$ be*

$$W(a_{ij}, \sigma_{ij}, \tau_{ij}; p, N-1, 2)$$

defined by (7). *Then the h-th moment of $|\, a_{ij}\,|$ is*

$$E(|\, \alpha_{ij}\,|^h) = |\, \sigma^{ij}\,|^h\, 2^{hp}\exp\left[-\tfrac{1}{2}\kappa_1^2+\kappa_2^2\right]\frac{\prod_{i=3}^{p}\Gamma(\frac{1}{2}[N-i]+h)}{\prod_{i=3}^{p}\Gamma(\frac{1}{2}[N-i])}$$

$$(39)\quad \sum_{\alpha,\beta_1,\beta_2=0}^{\infty} \frac{(\kappa_1^2)^{\alpha+\beta_1}(\kappa_2^2)^{\alpha+\beta_2}\Gamma(\tfrac{1}{2}[N-1]+h+\alpha+\beta_1)\Gamma(\tfrac{1}{2}[N-1]+h+\alpha+\beta_2)}{2^{2\alpha+\beta 1+\beta 2}\alpha!\beta_1!\beta_2!\Gamma(\tfrac{1}{2}[N-2]+\alpha)\Gamma(\tfrac{1}{2}[N-1]+2\alpha+\beta_1+\beta_2)} \times \frac{\Gamma(\tfrac{1}{2}[N-2]+h+\alpha)}{\Gamma(\tfrac{1}{2}[N-1]+h+\alpha)},$$

with κ_i^2 *defined by* (9).

The h-th moment of the generalized variance $|a_{ij}/(N-1)|$ is obtained by dividing the above expression by $(N-1)^{ph}$. This formula holds for all $h > -\frac{1}{2}(N-p)$.

6. The moments of the criterion for testing linear hypothesis in the linear and planar non-central cases.

6.1. *The moments of the criterion.* There are several linear hypotheses concerning the means of multivariate normal populations that can be included in a general formulation of the problem. We shall first of all consider a simple case of a linear hypothesis and find the moments of the criterion under linear and planar alternatives. In Section 6.2 we shall indicate some linear hypotheses that can be reduced to this simple case. Regression problems and the problem of equality of means in several populations (studied by Wilks) are included.

Suppose the variates $z_{i\alpha}$ $(i = 1, 2, \cdots, p;\ \alpha = 1, 2, \cdots, n)$ and $y_{i\gamma}$ $(i = 1, 2, \cdots p;\ \gamma = 1, 2, \cdots, q)$ have the probability element

$$\frac{|\sigma^{ij}|^{\frac{1}{2}(n+q)}}{(2\pi)^{\frac{1}{2}p(n+q)}} \exp\left[-\tfrac{1}{2}\sum_{i,j=1}^{p}\sum_{\alpha=1}^{n}\sigma^{ij}z_{i\alpha}z_{j\alpha}\right]$$

(40)

$$\exp\left[-\tfrac{1}{2}\sum_{i,j=1}^{p}\sum_{\gamma=1}^{q}\sigma^{ij}(y_{i\gamma}-\mu_{i\gamma})(y_{j\gamma}-\mu_{j\gamma})\right]\prod_{i=1}^{p}\prod_{\alpha=1}^{n}dz_{i\alpha}\prod_{i=1}^{p}\prod_{\gamma=1}^{q}dy_{i\gamma}.$$

Let us consider the hypothesis H_0 that the means of mp y's are zero, namely,

$$H_0:\ \mu_{i\gamma}=0 \quad (i=1,2,\cdots,p;\gamma=1,2,\cdots,m)$$

Let

$$(41)\quad a_{ij}=\sum_{\gamma=1}^{m}y_{i\gamma}\alpha_{j\gamma},$$

$$(42)\quad b_{ij}=\sum_{\alpha=1}^{n}z_{i\alpha}z_{j\alpha},$$

$$(43)\quad c_{ij}=a_{ij}+b_{ij}.$$

Then the likelihood ratio criterion for testing H_0, called by Hsu [8] the Wilks-Lawley hypothesis, is the $\frac{1}{2}(n+q)$ power of

$$(44)\quad W=\frac{|b_{ij}|}{|c_{ij}|}.$$

Under the null hypothesis the b_{ij} have a Wishart distribution with n degrees of freedom, and the a_{ij} are distributed independently of b_{ij} such that c_{ij} has a Wishart distribution with $n + m$ degrees of freedom. Wilks [5] has given the moments of W and in some special cases the distribution of W.

We shall now obtain the moments of W for distributions specified by (40) where the rank of $|| \mu_{i\gamma} ||$ ($\gamma = 1, 2, \cdots, m$) is 2, i.e., the planar case. Under this assumption the b_{ij} have a Wishart distribution with n degrees of freedom, the a_{ij} are independently distributed in such a way that the c_{ij} have a non-central Wishart distribution with $n + m$ degrees of freedom. Let κ_1^2 and κ_2^2 be the non-zero roots of

$$\left| \sum_{\gamma=1}^{m} \mu_{ij}\mu_{j\gamma} - \lambda\sigma_{ij} \right| = 0. \tag{45}$$

It is clear that the distribution of W is unchanged if σ^{ij} is set equal to δ_{ij}. Furthermore, we can take $\tau_{ij} = \kappa_i^2\delta_{ij}$ then the c_{ij} are distributed according to $W(c_{ij}, \delta_{ij}, \kappa_j^2\delta_{ij}; p, n, 2)$ with $n + m$ degrees of freedom. The moments will be obtained by a method similar to that used by Wilks [5].

Let the expected value given by (39) be

$$E(|c_{ij}|^h) = K(n + m, h, p, \kappa_i^2), \tag{46}$$

which is a constant depending on $n + m$, h, p, κ_1^2, and κ_2^2. If $D(a_{ij})$ represents the distribution function of the a_{ij}, one can write (46) as

$$K(n+m, h, p, \kappa_i^2) = \frac{1}{2^{\frac{1}{2}pn}\pi^{\frac{1}{4}p(p-1)}\prod_{i=1}^{p}\Gamma(\frac{1}{2}[n+1-i])} \tag{47}$$

$$\int |c_{ij}|^h |b_{ij}|^{\frac{1}{2}(n-p-1)} \exp[-\tfrac{1}{2}\textstyle\sum b_{ii}]\, D(a_{ij})] \prod_{i=1}^{p}\prod_{j=i}^{p} db_{ij}\, dA$$

where dA is the volume element of the a_{ij}, and where the integration is over the entire (permissable) ranges of the b_{ij} and a_{ij}. Equation (47) holds since the c's are functions of the b's and a's. Multiplying (47) by

$$2^{\frac{1}{2}pn}\prod_{i=1}^{p}\Gamma(\tfrac{1}{2}[n+1-i]), \tag{48}$$

then replacing n by $g + 2$ and dividing by (48) again, we obtain

$$K(n+m+2g, h, p, \kappa_i^2)\frac{2^{\frac{1}{2}p(n+2g)}\prod_{i=1}^{p}\Gamma(\frac{1}{2}[n+1-i]+g)}{2^{\frac{1}{2}pn}\prod_{i=1}^{p}\Gamma(\frac{1}{2}[n+1-i])} \tag{49}$$

$$= \frac{1}{2^{\frac{1}{2}pn}\pi^{\frac{1}{4}p(p-1)}\prod_{i=1}^{p}\Gamma(\frac{1}{2}[n+1-i])}$$

$$\cdot \int |c_{ij}|^h |b_{ij}|^{\frac{1}{2}(n+2g-p-1)} \exp\left[-\tfrac{1}{2}\sum_{i=1}^{p} b_{ii}]\, D(a_{ij}) \prod_{i=1}^{p}\prod_{i=j}^{p} db_{ij}\, dA\right].$$

By definition the right hand side of (49) is the expected value of $|c_{ij}|^h |b_{ij}|^g$. Hence

$$E(|c_{ij}|^h |b_{ij}|^g) = K(n+m+2g, h, p, \kappa_i^2) \frac{2^{pg} \prod_{i=1}^{p} \Gamma(\frac{1}{2}[n+1-i]+g)}{\prod_{i=1}^{p} \Gamma(\frac{1}{2}[n+1-i])}$$

In this expression it is permissible to set h equal to $-g$ (n could have been replaced by $n+2g$ in (47) to insure the argument of each Γ function being positive). Then we have

$$E(W^g) = E(|c_{ij}|^{-g} |b_{ij}|^g)$$

$$= K(n+m+2g, -g, p, \kappa_i^2) \frac{2^{gp} \prod_{i=1}^{p} \Gamma(\frac{1}{2}[n+1-i]+g)}{\prod_{i=1}^{p} \Gamma(\frac{1}{2}[n+1-i])}.$$

Finally, the g-th moment is

$$E(W^g) = \exp[-\tfrac{1}{2}(\kappa_1^2+\kappa_2^2)] \frac{\prod_{i=3}^{p} \Gamma(\frac{1}{2}[n+m+1-i]) \prod_{i=1}^{p} \Gamma(\frac{1}{2}[n+1-i]+g)}{\prod_{i=3}^{p} \Gamma(\frac{1}{2}[n+m+1-i]+g) \prod_{i=1}^{p} \Gamma(\frac{1}{2}[n+1-i])}$$

$$\cdot \sum_{\alpha,\beta_1,\beta_2=0}^{\infty} \left[\frac{(\kappa_1^2)^{\alpha+\beta_1} (\kappa_2^2)^{\alpha+\beta_2} \Gamma(\frac{1}{2}[n+m]+\alpha+\beta_1) \Gamma(\frac{1}{2}[n+m]+\alpha+\beta_2)}{2^{2\alpha+\beta_1+\beta_2} \alpha!\beta_1!\beta_2! \Gamma(\frac{1}{2}[n+m-1]+g+\alpha) \Gamma(\frac{1}{2}[n+m]+\alpha)} \right.$$

$$\left. \cdot \frac{\Gamma(\frac{1}{2}[n+m-1]+\alpha)}{\Gamma(\frac{1}{2}[n+m]+g+2\alpha+\beta_1+\beta_2)} \right]. \tag{50}$$

We can summarize in the following theorem:

THEOREM 6. *Let $z_{i\alpha}$ $(i = 1, 2, \cdots, p; \alpha = 1, 2, \cdots, n)$ and $y_{i\gamma}$ $(i = 1, 2, \cdots, p; \gamma = 1, 2, \cdots, q)$ have* (40) *as a joint distribution. Define a_{ij}, b_{ij} and c_{ij} by* (41), (42), *and* (43), *respectively. Let κ_1^2 and κ_2^2 be the non-zero roots of* (45). *Then the g-th moment of W, defined by* (44), *is* (50). Expression (50) gives the moments of W in the planar case. The linear case is a special case of the planar case, that is, it is the planar case for $\kappa_2^2 = 0$. The g-th moment of W in the linear case is given by

$$E(W^g) = \exp[-\tfrac{1}{2}\kappa_1^2] \frac{\prod_{i=2}^{p} \Gamma(\frac{1}{2}[n+m+1-i]) \prod_{i=1}^{p} \Gamma(\frac{1}{2}[n+1-i]+g)}{\prod_{i=2}^{p} \Gamma(\frac{1}{2}[n+m+1-i]+g) \prod_{i=1}^{p} \Gamma(\frac{1}{2}[n+1-i])}$$

$$\times \sum_{\beta_1=0}^{\infty} \frac{(\kappa_1^2)^{\beta_1} \Gamma(\frac{1}{2}[n+m]+\beta_1)}{2^{\beta_1} \beta_1! \Gamma(\frac{1}{2}[n+m]+g+\beta_1)}. \tag{51}$$

For $\kappa_1^2 = 0$, (51) reduces to the expresion given for the moments under the null hypothesis.

Wilks [7] has given the distribution of W under the null hypothesis for several special cases (i.e., certain pairs of n and p). In general, however, the distribution function is too complicated to write down explicitly. When the null hypothesis is not satisfied (i.e., at least one $\kappa_1^2 \neq 0$) the distribution functions are yet more involved. Hence, we shall not write any explicitly.

Hsu [8] has given the asymptotic distribution of W. Suppose that

$$\Psi_n = \sum_{i,j=1}^{p} \sum_{\gamma=1}^{m} \mu_{i\gamma}\mu_{j\gamma}$$

tends to the limit Ψ_0 as n tends to infinity (if the μ's are functions of n). Then the limiting distribution of $x = -(n+q)\log W$ (which equals $-2\log\Lambda$, where Λ is the likelihood ratio criterion) is

$$2^{-\frac{1}{2}pm} e^{-\frac{1}{2}\Psi_0} x^{\frac{1}{2}pm-1} e^{-\frac{1}{2}x^2} \sum_{\alpha=0}^{\infty} \frac{\Psi_0^\alpha x^\alpha}{2^{2\alpha}\alpha!\,\Gamma(\frac{1}{2}pm+\alpha)}. \tag{52}$$

That is, it is the χ'^2 distribution with pm degrees of freedom and parameter Ψ_0.

For most purposes, alternative hypotheses of the means being on a line (i.e., of rank one) are sufficiently general. In any particular case, one can compute from (51) mumerical values for several moments and then fit an appropriate distribution function. If one wishes to consider alternative hypotheses of rank two, one can use (50) and similarly compute numerical values for moments. The series in either (51) or (50) converge rapidly. To construct an approximate power function for linear alternatives, say, one would fit distribution functions for several values of κ_1^2 and find the desired percentage levels.

There is a matrix $\|\,d_{ij}\,\|$ such that

$$\|\,b_{ij}\,\| = \|\,d_{ij}\,\| \cdot \|\,d_{ij}\,\|'$$

and

$$\|\,a_{ij}\,\| = \|\,d_{ij}\,\| \cdot \|\,\lambda_j\delta_{ij}\,\| \cdot \|\,d_{ij}\,\|',$$

where the λ's are roots of

$$|\,a_{ij} - \lambda b_{ij}\,| = 0 \tag{53}$$

It follows that

$$\|\,c_{ij}\,\| = \|\,d_{ij}\,\| \cdot \|\,(1+\lambda_j)\delta_{ij}\,\| \cdot \|\,d_{ij}\,\|'.$$

Then W can be written as

$$W = \frac{|\,d_{ij}\,| \cdot |\,d_{ij}\,|'}{|\,d_{ij}\,| \cdot |\,(1+\lambda_j)\delta_{ij}\,| \cdot |\,d_{ij}\,|'}$$

$$= \frac{1}{\prod_{j=1}^{p}(1+\lambda_j)}. \tag{54}$$

The distribution of the roots of (53) in the linear case has been given by Roy [9] for a_{ij} o dimensionality p.[5] The distribution in the planar case has been indicated by Anderson [3]. One could obtain the probability of W not exceeding a given value by integrating the λ's over the proper range.

6.2. *Examples of the general linear hypothesis.* A number of hypotheses concerning the expected values of variates with multivariate normal distributions can be put into the form of H_0. The equivalence of the hypotheses is demonstrated by means of linear transformations.

As an example consider the hypothesis H_1 that the means of several normal multivariate populations are equal when the respective covariance matrices are equal. Let $x_{i\alpha}^u$ be the α-th ($a = 1, 2, \cdots, N^u$) observation of the i-th ($i = 1, 2, \cdots, p$) variate in the u-th ($u = 1, 2, \cdots, U$) population. Let

$$E(x_{i\alpha}^u) = \mu_i^u \qquad \begin{matrix}(i = 1, 2, \cdots, p)\\ (u = 1, 2, \cdots, U),\end{matrix} \tag{55}$$

and let the covariance matrix be $||\sigma_{ij}||$. Then the hypothesis is

$$H_1 : \mu_i^u = \mu_i \qquad \begin{matrix}(i = 1, 2, \cdots, p)\\ (u = 1, 2, \cdots, U).\end{matrix} \tag{56}$$

For testing this hypothesis let

$$b_{ij} = \sum_{u=1}^{U} \sum_{\alpha=1}^{N^u} (x_{i\alpha}^u - \bar{x}_i^u)(x_{j\alpha}^u - \bar{x}_j^u), \tag{57}$$

$$a_{ij} = \sum_{u=1}^{U} N^u(\bar{x}_i^u - \bar{x}_i)(\bar{x}_j^u - \bar{x}_j), \tag{58}$$

where

$$\bar{x}_i^u = \frac{1}{N^u} \sum_{\alpha=1}^{N^u} x_{i\alpha}^u, \tag{59}$$

$$\bar{x}_i = \frac{1}{N} \sum_{u=1}^{u} \sum_{\alpha=1}^{N^u} x_{i\alpha}^u,$$

$$N = \sum_{u=1}^{u} N^u.$$

The b_{ij} have $n = N - U$ degrees of freedom and $c_{ij} = a_{ij} + b_{ij}$, have $N - 1$ degrees of freedom. Then the $N/2$ root of the likelihood ratio criterion for H_1 is W defined by (44). For this case equation (45) is

$$\left| \sum_{u=1}^{u} N^u(\mu_i^u - \bar{\mu}_i)(\mu_j^u - \bar{\mu}_j) - \lambda\sigma_{ij} \right| = 0,$$

where

$$\bar{\mu}_i = \frac{1}{N} \sum_{u=1}^{u} N^u \mu_i^u.$$

[5] Roy erroneously claims his distribution to hold for the planar case and higher rank.

Hsu has demonstrated that the general regression problem can be put into the form of H_0. Suppose that $x_{i\alpha}$ $(i = 1, 2, \cdots, p;\ \alpha = 1, 2, \cdots, N)$ follow a multivariate normal distribution with covariance matrix $\| \sigma_{ij} \|$, and let the expected value of $x_{i\alpha}$ be

$$E(x_{i\alpha}) = \sum_{r=1}^{q} \beta_{ir} w_{r\alpha} \qquad (q \leq N - p),$$

where the q by N matrix

$$W = \| w_{r\alpha} \|$$

is of rank q. Let H_2 be the hypothesis that

$$H_2 : B_1 = \| \beta_{iu} \| = 0 \qquad (i = 1, 2, \cdots, p; u = 1, 2, \cdots, m \leq q)$$

with the w's known. Let

$$W_1 = \| w_{u\alpha} \| \qquad (u = 1, 2, \cdots, m; \alpha = 1, 2, \cdots, N)$$

$$W_2 = \| w_{r\alpha} \| \qquad (r = m + 1, \cdots, 9;\ \alpha = 1, 2, \cdots, N),$$

$$X = \| x_{i\alpha} \| \qquad (i = 1, 2, \cdots, p; \alpha = 1, 2, \cdots, N).$$

Let

$$\| b_{ij} \| = XX' - XW'(WW')^{-1}WX$$

$$\| c_{ij} \| = XX' - XW_2'(W_2W_2')^{-1}W_2X'.$$

(with $\| c_{ij} \| = XX'$ if $W_2 = 0$). Then the likelihood ratio criterion for H_2 is the $N/2$-th power of W, defined by (44).

The equation (45) can be written in terms of Z, B_1, and W as

$$| B_1W_1(I - W_2'(W_2W_2')^{-1}W_2)W_1'B_1' - \lambda\Sigma | = 0 \tag{60}$$

for $m < q$. If $m = q$, (45) becomes

$$| B_1WW'B_1' - \lambda\Sigma | = 0. \tag{61}$$

In (60) and (61) there are no more non-zero roots than the rank of B_1. It is clear that the roots of (60) (or (61)) depend on the matrix W as well as B_1. The distribution of Λ the likelihood ratio criterion under the null hypothesis does not depend on the distribution of the matrix W (if W is not constant). However, the distribution when the null hypothesis is not satisfied does depend on κ_1^2 or on κ_1^2 and κ_2^2, and hence, on the distribution of the elements of W as well as the value of B_1.

The special case of H_0 for $m = q = 1$ gives as the likelihood ratio criterion as a function of Hotelling's generalized T^2. From the moments indicated in (50) we can deduce the distribution of T^2 when the null hypothesis is not true [3]. This result has been obtained by Hsu [10] by another method.

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ON HOTELLING'S WEIGHING PROBLEM[1]

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1. Summary. The paper contains some solutions of the weighing problems proposed by Hotelling [1]. The experimental designs are applicable to a broad class of problems of measurement of similar objects. The chemical balance problem (in which objects may be placed in either of the two pans of the balance) is almost completely solved by means of designs constructed from Hadamard matrices. Designs are provided both for a balance which has a bias and for one which has no bias.

The spring balance problem (in which objects may be placed in only one pan) is completely solved when the balance is biased. For an unbiased spring balance, designs are given for small numbers of objects and weighing operations. Also the most efficient designs are found for the unbiased spring balance, but it is shown that in some cases these cannot be used unless the number of weighings is as large as the binomial coefficient $\binom{p}{\frac{1}{2}p}$ or $\binom{p}{\frac{1}{2}(p+1)}$ where p is the number of objects.

It is found that when p objects are weighed in $N \geq p$ weighings, the variances of the estimates of the weights are of the order of σ^2/N in the chemical balance case (σ^2 is the variance of a single weighing), and of the order of $4\sigma^2/N$ in the spring balance case.

2. Introduction. The problem is fully discussed by Hotelling [1] and refers to the design of a certain class of simple experiments. We may consider the typical example of the class to be that of weighing several small objects on a chemical balance or other weighing device. Hotelling and Yates [2] have shown that the individual weights may be determined more accurately by weighing the objects in combinations rather than weighing each one separately. The designs are applicable to a great variety of problems of measurement, not only of weights, but of lengths, voltages and resistances, concentrations of chemicals in solutions, in fact any measurements such that the measure of a combination is a known linear function of the separate measures with numerically equal coefficients. The designs should be particularly useful in biological and chemical laboratories engaged in routine chemical analyses. We shall, however, in the interest of simplicity, discuss the problem in the language of weighing operations.

A particular design is denoted by a matrix. The three objects to be weighed in four weighing operations may be weighed by the following design:

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$$X = \begin{Vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{Vmatrix}$$

where the rows refer to weighing operations and the columns refer to the objects. In the above design the first two objects are weighed together in the first weighing operation, the first and third objects are weighed together in the second weighing operation, etc. From the four resulting weights the individual weights are estimated by the method of least squares. The design problem consists of finding matrices which will minimize the variances of these estimates.

There are two distinct though closely related problems here. One is to find efficient designs for the case in which the measure of a combination can only be the sum of the individual measures. This would be the case, for example, in weighing objects with a spring balance and we shall refer to it as the spring balance problem. The other problem is to find designs when an individual measurement may be either added or subtracted in a combination. This would be the case in weighting objects with a chemical balance (since an object may be put in either pan of the balance) and will be called the chemical balance problem. In the latter problem the design matrix may contain 0's, 1's, and -1's, whereas in the spring balance problem the matrix may contain only 0's and 1's.

We shall use Hotelling's notation. There are p objects with weights b_1, b_2, $\cdots$, b_p to be weighed in $N \geq p$ weighing operations. The design matrix is denoted by

$$X = \| x_{\alpha i} \| \quad \alpha = 1, \cdots, N; i = 1, \cdots, p. \tag{1}$$

Denoting the transpose of X by X', let

$$X'X = \| a_{ij} \| = \| a^{ij} \|^{-1} \tag{2}$$

$$g_i = \sum_{\alpha} x_{\alpha i} y_{\alpha} \tag{3}$$

where y_α is the observed result of the α-th weighing operation. The least squares estimates of the b_i are

$$\hat{b}_i = \sum_j a^{ij} g_j \tag{4}$$

and the variances of these estimates are

$$\sigma_i^2 = a^{ii}\sigma^2 \tag{5}$$

where σ^2 is the error variance of a single weighing operation. The a^{ii} will be called variance factors.

Hotelling's main theorem states that or any design, $a^{ii} \geq 1/N$, hence the best possible design is one such the inverse of the product of the design matrix by its transpose has its main diagonal elements equal to $1/N$. We shall call such a design an optimum design. Examples show that optimum designs do not exist for all values of N and p.

When an optimum design does not exist, the question arises as to how a best design shall be defined. In the present paper a design will be called best if the determinant of the matrix $|| a^{ij} ||$ is minimized. A best design in this sense is, therefore, a design which gives the smallest confidence region in the $\hat{b}_i$ $(i = 1, 2, \cdots, p)$ space for the estimates of the weights.

In certain situations, other definitions of best designs may conceivably be preferred. Thus, problems may arise in which one might prefer:

(a) to minimize the variance factors subject to the restriction that they be equal, (b) to minimize some function of the variance factors, or (c) to minimize only a certain subset of the a^{ii} on a minor of the matrix $|| a^{ij} ||$ as might be the case when one wanted only rough estimates of the weights of some of the objects, but accurate estimates of the others.

When an optimum design exists, the confidence regions are not only minimized, but, as Hotelling has shown, the variance factors are also minimized. It is not true in general, however, that a best design as here defined (minimum confidence regions) will also minimize the variance factors. Examples illustrating this point are given in the last part of section 6 and the first part of section 7.

3. Hadamard Matrices. The problem of finding the best designs is closely related to the Hadamard determinant problem. Hadamard [3] proved the following result: If the elements $x_{\alpha\beta}$ of a square matrix X are restricted to the range $-1 \leq x_{\alpha\beta} \leq 1$, the maximum possible value of the determinant of X is $N^{\frac{1}{2}N}$, and when this maximum is achieved all $x_{\alpha\beta} = \pm 1$ and the matrix is orthogonal in the sense that $X'X$ is a diagonal matrix; the non-zero elements of $X'X$ are all equal to N. A matrix X which satisfies these conditions will be denoted by H_N. Obviously if H_N exists for a given N, it is the solution of the design problem in the chemical balance case when $N = p$.

With regard to the existence of H_N, it is known that a necessary condition is

$$N \equiv 0 \pmod 4$$

with the exception of $N = 2$. It is not known however whether the above condition is sufficient, although it is known (Paley [4]) that H_{4k} exists for the range

$$0 < 4k \leq 100$$

with the possible exception of $4k = 92$. Paley and Williamson [5] give methods of constructing H_{4k} in the given range (excepting 92) based on the theory of finite fields.

When N is a power of two, H_N is easily constructed by taking direct products of

$$H_2 = \begin{Vmatrix} 1 & 1 \\ 1 & -1 \end{Vmatrix}$$

Thus

$$H_4 = H_2 \cdot H_2 = \begin{Vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & +1 \end{Vmatrix}$$

Sylvester [6] first studied this class of matrices and Kishen [7] has described weighing designs based on this subset of the H_N.

The following examples of Hadamard matrices may be found in the literature: Paley [4] exhibits an H_8, H_{12}, and H_{28}: Kishen gives an H_{16}. From these examples H_{24} and H_{32} may be constructed at once from the direct products $H_2 \cdot H_{12}$ and $H_2 \cdot H_{16}$. The following is an H_{20}:

$$\left\|\begin{array}{cccccccccccccccccccc}
+ & - & - & - & - & + & - & - & - & - & + & + & - & - & + & + & - & + & + & - \\
- & + & - & - & - & - & + & - & - & - & + & + & + & - & - & - & + & - & + & + \\
- & - & + & - & - & - & - & + & - & - & - & + & + & + & - & + & - & + & - & + \\
- & - & - & + & - & - & - & - & + & - & - & - & + & + & + & + & + & - & + & - \\
- & - & - & - & + & - & - & - & - & + & + & - & - & + & + & - & + & + & - & + \\
- & + & + & + & + & + & - & - & - & - & - & + & - & - & + & + & + & - & - & + \\
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- & + & - & + & - & + & - & - & - & + & - & - & + & - & - & - & - & + & - & - \\
- & - & + & - & + & + & + & - & - & - & - & - & - & + & - & - & - & - & + & - \\
+ & - & - & + & - & - & + & + & - & - & - & - & - & - & + & - & - & - & - & +
\end{array}\right\|$$

where the signs represent ± 1. This example was constructed by Williamson's method [5]. Thus examples of H_{4k} for the range $4 \leq 4k \leq 32$ are immediately available and methods of construction exist for the range $36 \leq 4k \leq 88$.

4. Chemical Balance Problem. When $N = 0 \pmod 4$ an optimum design exists if H_N exists and is obtained by using any p columns of H_N. When $N \not\equiv 0 \pmod 4$ we may construct very efficient designs as follows: If $N \equiv 1$ we may add a row of ones to H_{N-1}; if $N \equiv 2$ we may add two rows of ones or a row of H_2's to H_{N-2}; and if $N \equiv 3$ we may delete one row from H_{N+1}. The worst of these designs will be obtained when two rows of ones are added to an H_{N-2}, and in this case the variance factors are

$$a^{ii} = \frac{1}{N-2}\,\frac{N+2p-4}{N+2p-2} < \frac{1}{N-2}. \tag{6}$$

Since it is known that these factors must be greater than $1/N$ for the best possible design in this case, the above design will be quite near the best design for large N.

For small values of N we shall consider only the case $N = p$, since if one

wanted to make $N > p$ weighings, he would normally choose N to be a multiple of four because the gain in efficiency by using optimum designs is rather large for small N. In general more than p weighings would be required because σ^2 is not usually known. Thus several additional weighings may be made in order to obtain several degrees of freedom for estimating σ^2.

When H_N does not exist we have already defined the best design as one which minimizes the confidence region for estimating the weights; that is equivalent to maximizing $|a_{ij}|$ or minimizing $|a^{ij}|$. There may be several designs with the same minimum, but we shall not give all of them. Thus when $p = 3$ the best designs are

$$X = \begin{Vmatrix} + & + & 0 \\ + & - & + \\ - & + & + \end{Vmatrix}, \quad \begin{Vmatrix} + & + & + \\ + & - & + \\ - & + & + \end{Vmatrix} \text{ and } \begin{Vmatrix} + & + & - \\ + & - & + \\ - & + & + \end{Vmatrix}$$

all of which have $A = 16$ (which is considerably smaller than the value 27 that A would have if an optimum design existed). Using the notation

$$(a^{ii}) = (a^{11}, a^{22}, \cdots, a^{pp}),$$

the first of the above designs for $p = 3$ gives

$$(a^{ii}) = (\tfrac{3}{8}, \tfrac{3}{8}, \tfrac{1}{2})$$

while the second and third give

$$(a^{ii}) = (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}).$$

For $N = p = 5$, two best designs are

$$X = \begin{Vmatrix} + & + & + & + & - \\ + & + & + & - & + \\ + & + & - & + & + \\ + & - & + & + & + \\ - & + & + & + & + \end{Vmatrix} \text{ and } \begin{Vmatrix} + & - & - & - & - \\ + & + & + & - & - \\ + & - & + & - & + \\ + & - & + & + & - \\ + & + & - & + & + \end{Vmatrix}$$

both of which have

$$A = 3^2 2^8 \text{ and } (a^{ii}) = (2/9, 2/9, 2/9, 2/9, 2/9)$$

For $N = p = 6$, a best design is

$$X = \begin{Vmatrix} + & - & - & - & - & - \\ + & - & - & - & + & + \\ + & - & - & + & + & - \\ + & - & + & + & - & + \\ + & + & + & - & + & - \\ + & + & - & + & - & + \end{Vmatrix}$$

which has

$$A = 5^2 2^{10} \text{ and all } a^{ii} = 1/5.$$

For $N = p = 7$, a best design is

$$X = \begin{Vmatrix} + & - & - & - & - & - & - \\ + & + & + & + & - & - & - \\ + & + & - & - & - & + & + \\ + & + & - & - & + & + & - \\ + & - & - & + & + & - & + \\ + & - & + & + & - & + & - \\ + & - & + & - & + & - & + \end{Vmatrix}$$

which has

$$A = 2^{12}3^4 \text{ and all } a^{ii} = 1/6.$$

These designs were constructed by a method due to Williamson [8] which will be described in sections 5 and 7. It is interesting to note that no minor of an H_8 is a best design for $N = p = 7$, for any minor of an H_8 gives $A = 2^{13} < 2^{12}3^4$ and all $a^{ii} = \frac{1}{4}$.

5. Spring Balance Problem. $N = p = 4k + 3$. When $N = p$ and $N \equiv 3 \pmod 4$ the best possible design for the spring balance case is determined by H_{N+1} if it exists. Let K_{N+1} denote a matrix formed from H_{N+1} by adding or subtracting the elements of the first row of H_{N+1} from the corresponding elements of the other rows in such a way as to make the first element of each of the remaining rows zero. Obviously

$$|K_{N+1}| = \pm |H_{N+1}|.$$

and excepting the first row, the elements of K_{N+1} are 0 and ± 2 with the signs of the non-zero elements the same for elements in the same row. Let L_N be the matrix obtained by omitting the first row and column of K_{N+1}, by changing all non-zero elements to $+1$, and by permitting two rows if necessary to make the determinant of L_N positive. Then

$$|H_{N+1}| = 2^N |L_N|$$

and it is clear that, given L_N, one could reverse the procedure and determine an H_{N+1}. In the same manner, there is a correspondence in general between square matrices with elements ± 1 and square matrices of one less order with elements 0 and 1. The ratio of the values of corresponding determinants is always 2^N if their determinants do not vanish; hence the 0,1 determinant will have its maximum value when its corresponding $+1$ determinant has a maximum value. Thus $|L_N|$ is the maximum value possible for a determinant of 0's and 1's of order N, and the value is

$$|L_N| = (N + 1)^{\frac{1}{2}(N+1)}/2^N. \tag{7}$$

The variance factors are

$$a^{ii} = 4N/(N + 1)^2.$$

We knew in advance, of course, that the a^{ii} would be greater than $1/N$ since an optimum design cannot exist unless the design matrix has its elements equal to ± 1, and we must here restrict the design to have only 0 and 1 as its elements. Since L_N is a best possible design for the spring balance case, it follows that designs for the spring balance problem can be no more than about $\frac{1}{4}$ as efficient as designs for the chemical balance problem.

6. Spring Balance $N > p$. When $N > p$ the device used in the chemical balance case to get optimum designs cannot be used. For if we select p columns from an L_N we may get rows of zeros which would waste weighing operations. A different approach is necessary and a clue is given by the designs L_N. In these designs p is odd and the objects are weighed $\frac{1}{2}(p + 1)$ at a time in each weighing operation. We shall show in general that objects should be weighed $\frac{1}{2}(p + 1)$ at a time when p is odd, and we shall obtain a corresponding result for p even.

Let P_r be a matrix whose rows are all the arrangements of r ones and $p - r$ zeros ($0 \leq r \leq p$). (The symbol should also have a subscript p but that is omitted because any specific value for p will always be clear from the context.) The matrix will have p columns and $\binom{p}{r}$ rows. Let Q be a matrix made up of matrices P_r arranged in vertical order. Let n_r be the number of times P_r is used in constructing Q. Q is a weighing design for p objects and

$$N = \sum_r n_r \binom{p}{r}$$

weighing operations. The matrix $Q'Q$ will have diagonal elements

$$a = \sum n_r \binom{p-1}{r-1} \tag{9}$$

and non-diagonal elements

$$b = \sum n_r \binom{p-2}{r-2}. \tag{10}$$

The determinant of $Q'Q$ is

$$A = (a - b)^{p-1}[a + (p - 1)b]$$

and we may write A in the form

$$A = c^{p-1}d$$

where

$$c = a - b, \text{ and } d = a + (p - 1)b. \tag{11}$$

We shall prove the following theorem:

If $p = 2k - 1$ *where* k *is a positive integer, and if* N *contains the factor* $\binom{p}{k}$, *then* A *will be maximized when* $n_k = N \Big/ \binom{p}{k}$ *and all other* $n_r = 0$.

We shall demonstrate this statement by showing that if any n_s $(s \neq k)$ is decreased and n_k is increased in such a way that N remains unchanged, then A will be increased. Let n_s be reduced by an amount m so chosen that

$$m' = m \binom{p}{s} \Big/ \binom{p}{k}$$

is an integer; we may then increase n_k by m' leaving N unchanged. It is readily found that these changes in n_s and n_k produce the following changes in c and d:

$$\Delta c = m \binom{p}{s} \frac{(k - s)(k - s - 1)}{p(p - 1)}$$

$$\Delta d = m \binom{p}{s} \frac{(k - s)(k + s)}{p}$$

both of which are positive on zero when $s < k$ and A is necessarily increased.

When $s > k$, Δc is positive but Δd is negative and it must be shown that the net effect of these changes is to increase A, we shall assume now that $n_r = 0$ when $r < k$.

$$\Delta A = (c + \Delta c)^{p-1}(d + \Delta d) - c^{p-1}d < [c^{p-1} + (p - 1)c^{p-2}\Delta c](d + \Delta d)$$
$$- c^{p-1}d < c^{p-2}[c\Delta d + (p - 1)d\Delta c + (p - 1)\Delta c\Delta d]$$

where in the second line we have omitted terms in Δc of higher order than the first. These terms are all positive since all their fctors are positive. The bracket in the last expression on substituting from (9), (10), and (11), may be reduced to

$$m \binom{p}{s}\Bigg[(k - s) \sum_{r \geq k} n_r \left\{\binom{p - 2}{r - 1} + (k - s - 1)\binom{p - 2}{r - 1}\right.$$
$$\left. + \frac{m}{p^2}\binom{p}{s}(k - s)^2(k + s)(k - s - 1)\right],$$

and then to

$$m \binom{p}{s}\Bigg[\sum_{r \geq k} n_r \binom{p - 1}{r - 1}(k - s) \frac{r(k - s) + (k + s + 1 - 2r)}{p - 1}$$
$$+ \frac{m}{p^2}\binom{p}{s}(k - s)^2(k + s)(k - s - 1)\Bigg].$$

Each term of the sum in the bracket is greater than or equal to zero when $k > 1$, $r \geq k$, $s > k$ since the fraction is readily seen to be negative or zero under these

circumstances. The fraction vanishes only when $k = 2$, $r = k$, $s = k + 1$. The other term in the bracket is negative but it is dominated by the term in the sum for which $r = s$, as may be shown as follows: The two terms in question may be written

$$n_s\binom{p-1}{s-1}(k-s)\frac{s(k-s)+k-s+1}{p-1} + \frac{m}{p}\binom{p-1}{s-1}\frac{(k-s)^2(k+s)(k-s-1)}{s}$$

and since $n_s \geq m$, this expression is less than or equal to

$$m\binom{p-1}{s-1}(k-s)\left[\frac{s(k-s)+k-s+1}{p-1} + \frac{(k^2-s^2)(k-s-1)}{ps}\right]$$

which is positive for $s > k$ since the bracket is negative as may be seen by factoring out $\dfrac{1}{p(p-1)s}$ and putting the result in the form

$$(k-s+1)(s^2+(p-1)k^2) - pk(p-s) + (2s+1)(k-s).$$

Thus ΔA has been shown to be positive and the theorem is proved.

The above argument has shown that P_k or repetitions of P_k give more efficient designs than any other combination of the designs $P_1, P_2, \cdots, P_k$. The question now arises as to whether these are the best possible designs. We shall show that they are by considering the matrices L_N of section 5 which are known to give the greatest efficiency in the spring balance case. Let $p = 4t + 3$ and let $N = \binom{p}{2t+2}$, and suppose L_p exists (i.e. H_{p+1} exists). Using P_{2t} as the weighing design we find the a_{ii} are

$$a_{ii} = 2N(t+1)/p$$
$$a_{ij} = N(t+1)/p \qquad i \neq j.$$

A single application of the design L_p gives

$$a_{ii} = 2(t+1)$$
$$a_{ij} = t+1 \qquad i \neq j$$

and N/p repetitions of L_p gives an a_{ij} matrix with elements equal to N/p times the given elements for one application of the design. The two designs are therefore equivalent and P_{2t} is a best design.

The variance factors for repetitions of the design P_k are

$$a^{ii} = \frac{4}{N}\frac{p^2}{(p-1)^2} \qquad N \equiv 0 \text{ Mod } \binom{p}{k} \tag{12}$$

and these are minimum variance factors[2] as may be shown by an argument entirely analogous to that used in proving the theorem. Thus P_k is a design which not only minimizes the confidence region for estimating the weights, but also minimizes the individual variance factors.

Efficient sub-matrices of the P_k have not been studied except for small p, but we may point out that square sub-matrices of order p which are as efficient as P_k do not exist unless H_{p-1} exists, for by the argument of section 4, it is possible to construct H_{p+1} from such sub-matrices. Hence we cannot obtain variance factors as small as those given by equation (12) when $N = p$ unless H_{p+1} exists.

The situation here is analogous to that in the chemical balance case. By a proper selection of N we can obtain a design with the maximum possible efficiency for any odd value of p. But here we are much more restricted in our choice of N. In the chemical balance case N could be any multiple of 4 for which an H_N existed; in the present case N must be a multiple of p even in the most favorable instance ($p = 4t + 3$), and for some values of p it may be necessary that N be a multiple of $\binom{p}{\frac{1}{2}(p+1)}$.

We now turn to the case in which p is even. The theorem corresponding to the one given at the beginning of this section is:

If $p = 2k$ where k is a positive integer, and if N contains the factor $\binom{p+1}{k+1}$, then A will be maximimized when

$$n_k = n_{k+1} = N \Big/ \binom{p+1}{k+1}$$

and all other $n_r = 0$.

We shall not prove this theorem in detail. By arguments analogous to those used earlier, it may be shown that A is increased when either n_s ($s < k$) is decreased and n_k is increased, or n_s ($s > k + 1$) is decreased and n_{k+1} is increased with N fixed. This done, we may put all $n_r = 0$ except n_k and n_{k+1} and then maximize A with respect to these two variables subject to the condition that

$$n_k \binom{p}{k} + n_{k+1} \binom{p}{k+1} = N.$$

The values of n_k and n_{k+1} which maximize A may be found by treating them as continuous variables and using the calculus.

The variance factors for these designs are

$$a^{ii} = \frac{4}{N} \frac{p}{p+2} \qquad N \equiv 0 \bmod \binom{p+1}{k+1}, \tag{13}$$

[2] The author is indebted to a referee for suggesting this property of the design, and for several other valuable suggestions and corrections to the paper.

but these are not minimum variance factors. In fact one can obtain smaller variance factors than these by using only P_k in the design (omitting P_{k+1} entirely). In this case

$$a^{ii} = \frac{4}{N}\frac{(p-1)^2+1}{p^2} \qquad N \equiv 0 \bmod \binom{p}{k} \tag{14}$$

and

$$\frac{(p-1)^2+1}{p^2} < \frac{p}{p+2} \quad \text{when} \quad p > 2.$$

We have not found explicitly the design which minimizes the variance factors for p even, but it appears that the design would be made up largely from P_k with a small proportion of the design devoted to P_{k+1}. Thus (14) is very nearly the minimum possible variance factor.

7. Spring Balance Designs for Small p. When $p = 2$, each object may be weighed r times by itself, and the two objects may be weighed together s times to give

$$\| a_{ij} \| = \begin{Vmatrix} r+s & s \\ s & r+s \end{Vmatrix}$$

and if A is maximized subject to $2r + s = N$ we find

$$r = s = N/3$$

$$a^{ii} = 2/N$$

provided N is a multiple of 3. The most efficient basic design is therefore

$$X = \begin{Vmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{Vmatrix}$$

in accordance with the previous section. When N is not a multiple of 3 the best design is obtained by using the first row of X for the odd weighing when $N = 3t + 1$, and the last two rows when $N = 3t + 2$.

The case $p = 2$ is notable in that there is almost nothing to be gained by weighing the objects in combination. For the variance factors $2/N$ would be obtained by simply weighing each object separately $N/2$ times. The advantage of weighing in combination is only that square confidence regions in the b_1, b_2 space are replaced by ellipses with somewhat smaller area. If $a^{ii} = (r + s)/(r^2 + 2rs)$ is minimized subject to $2r + s = N$, we find

$$r = N(3 - \sqrt{3})/3, \; a^{ii} = 1.866/N$$

so that the a^{ii} are reduced slightly from $2/N$ but at the expense of increasing the area of the elliptical confidence regions.

For $p = 3$ the most efficient design when $N = 3$ is

$$X = \begin{Vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{Vmatrix}$$

as given by L_3 or P_2. It is easily shown that for $N > 3$, the most efficient design is given by repeating X even when $N \not\equiv 0 \pmod 3$. Thus for $N = 4$ we would repeat one row of X, for $N = 5$ we would repeat two rows of X, and so forth. The variance factors are

$$\begin{aligned} a^{ii} &= \frac{9}{4N} && N = 3t \\ &= \frac{9(N+1)}{4(N-1)(N+2)} && N = 3t+1 \\ &= \frac{9(N+1)}{4(N-2)(N+1)} && N = 3t+2. \end{aligned}$$

For $p = 4$ we may attempt to find by trial and error a sub-matrix of the design given by using P_2 once and P_3 once, but this would be a tedious process and the labor would soon become prohibitive for larger values of p. Hence another method must be found for obtaining the best designs when $N = p$ except when L_p exists. A method is provided by Williamson [8]. Let D_p be the best design for $N = p$. Williamson shows that when $p < 7$, D_{p-1} is a minor of D_p, hence D_p may be found by adding a row and column of variables to D_{p-1} and expanding the determinant of the result by the bordered expansion. For small values of p it is easy to determine by inspection what values the variables should have in order to maximize the resulting expansion. Williamson determined D_4 and D_5 by this method.

There are two types of D_4 which give a maximum value of $A = 9$

$$D_4 = \begin{Vmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{Vmatrix} \text{ and } \begin{Vmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{Vmatrix}.$$

The variance factors are all 7/9 for the first of these, and for the second

$$(a^{ii}) = (7/9, 7/9, 7/9, 4/9).$$

When $N = 5$, $p = 4$, there are a number of designs which give a maximum A of 19. None of these however has all a^{ii} equal, and we shall give only one example:

$$X = \begin{Vmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{Vmatrix}$$

which has

$$(a^{ii}) = \left(\frac{12}{19}, \frac{12}{19}, \frac{13}{19}, \frac{8}{19}\right).$$

When $N = 6$, there appears to be no design superior to P_2. It has variance factors all equal to $5/12$ and $A = 48$,—a very large gain in efficiency over $N = 5$ at the expense of one additional observation.

When $p = 5$ there are three types of D_5 which give A a maximum value of 25, none of which has all variance factors equal. An example is

$$D_5 = \begin{Vmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{Vmatrix}$$

with

$$(a^{ii}) = \left(\frac{19}{25}, \frac{19}{25}, \frac{16}{25}, \frac{11}{25}, \frac{16}{25}\right).$$

For $p = 6$, an example of a D_6 with all a^{ii} equal which maximizes A is

$$D_6 = \begin{Vmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{Vmatrix}$$

with $A = 81$ and $a^{ii} = 17/27$. This example was constructed by the bordered expansion method from D_5 and it turns out to be a sub-matrix of P_3. It is not as efficient as P_3, however, since substitution of $N = p = 6$ in equation (14) gives $a^{ii} = 13/27$. Hence we have shown that there does not exist a minor of P_3 (for $p = 6$) of order 6 which is as efficient as P_3 itself.

For $p = 7$, there is a most efficient design given by L_7.

$$L_7 = \begin{Vmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{Vmatrix}$$

with $A = 2^{10}$ and all $a^{ii} = 7/16$.

D_p for $p = 8$, 9, and 10 could presumably be constructed from L_7 in the same way and the designs for $p = 4$, 5, and 6 were constructed from L_3, but the computations become very tedious for these larger values of p.

The designs given in section 3 were constructed from the above designs by the method described in section 4.

8. Bias in Measuring Devices. In some kinds of experiments it may be necessary to estimate a bias in the measuring scale in order to estimate the measures of the objects. Such a bias may simply be regarded as an additional object to be measured except that it is an object which must be included in all the measuring operations. In the chemical balance case the bias presents no difficulty, for if an H_N exists, then there exists an H_N with a column whose elements are all $+1$. Such an H_N may be constructed from any given H_N by merely changing the signs of all elements in rows which begin with a minus sign. The result will be an H_N with $+1$'s in the first column and that column may be assigned to the bias. We note that the gain in efficiency by measuring objects in combinations is even greater in the case of a biased measuring scale than when there is no bias. For if the objects were measured separately, their measures would be estimated by the difference of two scale readings and would have variance $2\sigma^2$; hence the variance factors a^{ii} are to be compared with 2 (rather than 1) in the case of bias.

In the spring balance case, the additional restriction that all the elements of one column be one necessarily reduces the efficiency of the designs in the sense that the variance factors for p objects and a bias will be larger than the variance factors for $p + 1$ objects without bias. When the measures of p objects and a bias are to be estimated from $N = p + 1$ measuring operations, a best design may be obtained by adding a row of zeros and a column of ones (in that order) to the best design for $N = p$ without bias. This can be seen by recalling that there are two determinantal exressions for the volume of a simplex with one vertex at the origin in a Euclidean p space. (A simplex (Sommerville, [9]) is a polytope with $p + 1$ vertices bounded by $p + 1$ $(p - 1)$-dimensional hyperplanes.) The determinant of the best design for $N = p$ (without bias) is proportional to the volume of the largest simplex with one vertex at the origin and the other vertices restricted to be selected from the vertices of the unit cube. A determinant of order $p + 1$ with a column of ones and the other elements zero or one also gives the volume of a simplex with vertices selected from the vertices of the unit cube. Hence the two determinants (one of order p and one of order $p + 1$) must have the same maximum value, and as one of the vertices may be selected arbitrarily in the case of bias, we may select the origin.

In general, for $N \geq p$, similar geometrical reasoning will show that the best designs for the spring balance problem in the case of bias are easily constructed from Hadamard matrices as described in the following theorem:

If X is a best design for the chemical balance problem in the case of bias and if X contains a row of $+1$'s, then a best design for the spring balance problem in the case of bias is given by replacing the -1's in X by zeros.

We have seen that the best design in the chemical balance case is obtained from a Hadamard matrix with a column of $+1$'s. Obviously the matrix may be also made to contain a row of $+1$'s by changing the signs of certain columns. The design X consists of the column of ones together with any other p columns. The determinant of $X'X$ is $1/p!^2$ times the sum of squares of the volumes of a set of simplexes in a p space. There are $\binom{N}{p+1}$ of these simplexes deter-

mined by the different combinations of the rows of X taken $p + 1$ at a time, and the coordinates of their vertices are the last p elements of the rows of X. The vertices are therefore selected from the vertices of a cube in the p space which has its edges parallel to the coordinate axes, the origin at its center, and the lengths of its edges equal to two. Since X is a best design, the vertices are selected so as to maximize the sum of squares of the volumes of the simplexes. Now in the spring balance case we must maximize the sum of squares of the volumes of a set of simplexes which have their vertices selected from the vertices of the unit cube. Obviously this may be done by selecting vertices corresponding to the selection given by X. Thus it is necessary only to set up a correspondence beteen the vertices of the two cubes. Since X contains the vertex $(1, 1, 1, 1, \cdots, 1)$ which is common to both cubes, the natural correspondence which identifies a vertex such as $(1, -1, -1, 1, -1, 1, \cdots)$ with $(1, 0, 0, 1, 0, 1, \cdots)$ may be used.

The variance factors for these spring balance designs are $4/N$ (for any $p \leq N$) when N is a multiple of four and H_N exists; when N is not a multiple of four and modifications of H_N as described in section 3 are used, the variance factors will differ from $4/N$ by terms of order $1/N^2$.

9. Addendum. After this paper was written, the paper of Plackett and Burman on "The Design of Multifactorial Experiments" appeared in *Biometrika*. Volume 33 (1946), pages 305–325. A part of this paper discusses Hadamard matrices much more completely than we have done in section 3. In particular Plackett and Burman have constructed all Hadamard matrices of order less than or equal to 100 (excepting 92).

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THE APPROXIMATE DISTRIBUTION OF STUDENT'S STATISTIC

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Summary. It is well known that various statistics of a large sample (of size n) are approximately distributed according to the normal law. The asymptotic expansion of the distribution of the statistic in a series of powers of $n^{-\frac{1}{2}}$ with a remainder term gives the accuracy of the approximation. H. Cramér [1] first obtained the asymptotic expansion of the mean, and recently P. L. Hsu [2] has obtained that of the variance of a sample. In the present paper we extend the Cramér-Hsu method to Student's statistic. The theorem proved states essentially that if the population distribution is non-singular and if the existence of a sufficient number of moments is assumed, then an asymptotic expansion can be obtained with the appropriate remainder. The first four terms of the expansion are exhibited in formula (35).

1. In a fundamental paper[1] P. L. Hsu [2] has devised a method for obtaining the asymptotic expansion of the distribution of various statistics. The present paper deals with the so-called Student statistic.

Let

$$\xi_1, \xi_2, \cdots, \xi_n$$

be n independent random variables having the same probability distribution represented by a distribution function $P(x)$. The rth moment and rth absolute moment are denoted by α_r and β_r respectively. It is assumed that $\alpha_1 = 0$ and that for a certain $k \geqq 3$, $\beta_k < \infty$ and that $\alpha_2 > 0$. Hence there is no loss of generality in assuming that $\alpha_2 = 1$.

Student's statistic is defined as

$$\bar{\xi}\left(\frac{\sum_{r=1}^{n}(\xi_r - \bar{\xi})^2}{n(n-1)}\right)^{-\frac{1}{2}} \quad \text{where} \quad \bar{\xi} = \frac{1}{n}\sum_{r=1}^{n}\xi_r .$$

For brevity, we consider

$$n\bar{\xi}\left(\sum_{r=1}^{n}(\xi_r - \bar{\xi})^2\right)^{-\frac{1}{2}}.$$

Let its distribution function be denoted by $F(z)$, i.e.,

$$Pr\left\{n\bar{\xi}\left(\sum_{r=1}^{n}(\xi_r - \bar{\xi})^2\right)^{-\frac{1}{2}} \leqq z\right\} = F(z).$$

[1] The definitions of the various constants A, A_k, Q_k, Λ_k, ϑ, Θ, Θ_k, are the same as in Hsu's paper.

Discarding the case $k = 3$ where we can prove a more precise result and the singular case which can be shown to admit no asymptotic expansion in the sense of Cramér [1], we shall prove in this paper the following theorem:

THEOREM. *Let $P(x)$ be non-singular and $\alpha_{2k} < \infty$ for some integer $k \geqq 4$. Then*

$$(1) \qquad F(z) = \Phi(z) + \chi(z) + R(z), \qquad \Phi(z) = \frac{1}{\sqrt{2\pi}} \int_0^z e^{-\frac{1}{2}y^2}\, dy,$$

where $\chi(z)$ is a linear combination of the derivatives $\Phi'(z), \cdots, \Phi^{(3k-10)}(z)$ with each coefficient of the form $n^{-\frac{1}{2}\nu}(1 \leqq \nu \leqq k-3)$ times a quantity depending only on $\alpha_3, \cdots, \alpha_{k-1}$ whose beginning terms are given in (35) *and where*

$$(2) \qquad |R(z)| \leqq Q_k(1 + |z|^{2k-3})n^{-\alpha_0}, \qquad \alpha_0 = \frac{(k-1)\left[\frac{k}{2}\right]}{2\left(\left[\frac{k}{2}\right]+1\right)},$$

where Q_k is a constant depending on k and $P(x)$.[1]

We shall need some of Hsu's lemmas, i.e., his lemma 3, lemma 7 (both for the particular case $m = 2$) and lemma 8. These we shall quote with this numbering. The application of Hsu's method to Student's statistic depends on the following lemma.

2. LEMMA A. *For $u \geqq -1$, $l \geqq 1$, we have*

$$1 + \sum_{j=1}^{2l-1} \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}-j\right)\Gamma(j+1)}\, u^j - \left(1 + \sum_{j=1}^{2l-1} \frac{(-1)^j \Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}-j\right)\Gamma(j+1)}\right) u^{2l}$$

$$\leqq \sqrt{1+u} \leqq 1 + \sum_{j=1}^{2l-1} \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}-j\right)\Gamma(j+1)}\, u^j.$$

PROOF. By Taylor's expansion of $\sqrt{1+u}$, we have

$$\sqrt{1+u} = 1 + \sum_{j=1}^{2l-1} \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}-j\right)\Gamma(j+1)}\, u^j + \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}-2l\right)\Gamma(2l+1)} (1+\vartheta u)^{-\frac{1}{2}(4l-1)} u^{2l},$$

whence it follows that $(1 + \vartheta u)^{-\frac{1}{2}(4l-1)}$ is finite, and positive. The right-hand side inequality follows.

Similarly, if $u \geqq 0$,

$$\sqrt{1+u} \geqq 1 + \sum_{j=1}^{2l-1} \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}-j\right)\Gamma(j+1)} u^j + \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}-2l\right)\Gamma(2l+1)} u^{2l}$$

$$\geqq 1 + \sum_{j=1}^{2l-1} \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}-j\right)\Gamma(j+1)} u^j - \left(1 + \sum_{j=1}^{2l-1} \frac{(-1)^j \Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}-j\right)\Gamma(j+1)}\right) u^{2l}$$

since by a well-known result on the binomial theorem we have

$$1 + \sum_{j=1}^{\infty} \frac{(-1)^j \Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}-j\right)\Gamma(j+1)} = \sqrt{1-1} = 0.$$

For $-1 \leqq u < 0$, we have

$$1 + \sum_{j=1}^{2l-1} \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}-j\right)\Gamma(j+1)} u^j - \sqrt{1+u} = \frac{N}{D}, \quad \text{say.}$$

For $-1 \leqq u < 0$, we have

$$D = 1 + \sum_{j=1}^{2l-1} \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}-j\right)\Gamma(j+1)} u^j + \sqrt{1+u}$$

$$\geqq 1 + \sum_{j=1}^{2l-1} \frac{(-1)^j \Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}-j\right)\Gamma(j+1)}.$$

Next,

$$N = \left(1 + \sum_{j=1}^{2l-1} \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}-j\right)\Gamma(j+1)} u^j\right)^2 - (1+u)$$

is a polynomial in u of the form

$$u^{2l}(a_0 + a_1 u + \cdots + a_{2l} u^{2l})$$

where $a_0 > 0$ and the successive coefficients have alternating signs; hence for $-1 \leq u \leqq 0$, $a_0 + a_1 u + \cdots + a_{2l} u^{2l}$ assumes its maximum at $u = -1$. This

maximum is obtained by putting $u = -1$ in the numerator, hence for $-1 \leqq u < 0$,

$$N \leqq u^{2l}\left(1 + \sum_{j=1}^{2l-1} \frac{(-1)^j \Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2} - j\right)\Gamma(j+1)}\right)^2$$

The left-hand side inequality in the lemma now follows.

For brevity we write the inequalities as

$$(3)\quad 1 + P_{2l}(u) = 1 + P_{2l-1}(u) - b_{2l}u^{2l} \leqq \sqrt{1+u} \leqq 1 + P_{2l-1}(u), \quad b_{2l} > 0.$$

3. We write

$$\sum_{r=1}^{n}(\xi_r - \bar{\xi})^2 = \sum_{r=1}^{n}\xi_r^2 - n\bar{\xi}^2 = n + \sqrt{n(\alpha_4 - 1)}\, X - Y^2,$$

where

$$X = \sum_{r=1}^{n} \frac{\xi_r^2 - 1}{\sqrt{n(\alpha_4 - 1)}}, \quad Y = \sqrt{n}\bar{\xi}.$$

Then Student's statistic may be written as

$$n\bar{\xi}\left(\sum_{r=1}^{n}(\xi_r - \bar{\xi})^2\right)^{-\frac{1}{2}} = Y\left(1 + \sqrt{\frac{\alpha_4 - 1}{n}}\, X - \frac{Y^2}{n}\right)^{-\frac{1}{2}}.$$

Then, for every z, we have

$$F(z) = Pr\left\{Y\left(1 + \sqrt{\frac{\alpha_4 - 1}{n}}\, X - Y^2\right)^{-\frac{1}{2}} \leqq z\right\}$$

$$= Pr\left\{\sqrt{1 + \frac{z^2}{n}}\, Y \leqq z\sqrt{1 + \sqrt{\frac{\alpha_4 - 1}{n}}\, X}\right\}.$$

For brevity let

$$\sqrt{1 + \frac{z^2}{n}}\, Y = V, \qquad \sqrt{\frac{\alpha_4 - 1}{n}}\, X = U.$$

Suppose $z \leqq 0$; then we have by (3),

$$z + zP_{2l-1}(U) \leqq z\sqrt{1+U} \leqq z + zP_{2l}(U)$$

$$(4)\qquad Pr\{V \leqq z + zP_{2l-1}(U)\} \leqq F(z) \leqq Pr\{V \leqq z + zP_{2l}(U)\}$$

Suppose $z > 0$; then we have by Lemma A a similar inequality with the extreme terms exchanged.

Now we take $l = \left[\frac{k}{2}\right]$, and fix it henceforth.

Our next step is to obtain an asymptotic expansion for

$$Pr\{V \leqq z + zP_m(U)\} = Pr\left\{Y \leq z\left(1+\frac{z^2}{n}\right)^{-\frac{1}{2}} + z\left(1+\frac{z^2}{n}\right)^{-\frac{1}{2}} P_m\left(\sqrt{\frac{\alpha_4-1}{n}}\,X\right)\right\}$$

with $m = 2l - 1$ or $2l$, $l \geqq 1$.

Let b be any real number, and

$$z\left(1+\frac{z^2}{n}\right)^{-\frac{1}{2}} P_m\left(\sqrt{\frac{\alpha_4-1}{n}}\,X\right) = L_m(X).$$

Until section 12, we shall write simply $L(x)$ for either of the $L_m(x)$.

4. Let W be the probability function of the distribution of the random point (X, Y) and let $f(t_1, t_2)$ be the characteristic function.

$$W(S) = Pr\{(X, Y) \epsilon S\} \quad \text{for every Borel set } S \text{ in } \mathfrak{R}_2$$

$$f(t_1, t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{it_1x+it_2y}\,dW = \left\{p\left(\frac{t_1}{\sqrt{n}}, \frac{t_2}{\sqrt{n}}\right)\right\}^n$$

$$p(t_1, t_2) = \int_{-\infty}^{\infty} e^{it_1(\alpha_4-1)^{-\frac{1}{2}}(x^2-1)+it_2x}\,dP.$$

Then

$$(5) \qquad Pr\{Y \leqq b + L(X)\} = \iint\limits_{y \leqq b+L(x)} dW = \iint\limits_{y \leqq b} dW + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(x, y)\,dW$$

where

$$G(x, y) = \begin{cases} 1 & \text{if } b < y \leqq b + L(x), \\ -1 & \text{if } b + L(x) < y \leqq b, \\ 0 & \text{otherwise} \end{cases}$$

We approximate $G(x, y)$ by $H(x, y)$, where

$$H(x, y) = \begin{cases} e^{-\epsilon x^2 l} & \text{if } b < y \leqq b + L(x) \\ -e^{-\epsilon x^2 l} & \text{if } b + L(x) < y \leqq b \\ 0 & \text{otherwise} \end{cases}$$

We approximate dW by $(w(x, y) + \gamma(x, y))\,dx\,dy$, where

$$w(x,y) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-it_1x-it_2y}\phi(t_1,t_2)\,dt_1\,dt_2$$

$$\gamma(x,y) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-it_1x-it_2y}\phi(t_1,t_2)\psi(it_1,it_2)\,dt_1\,dt_2$$

$$\phi(t_1,t_2) = e^{-\frac{1}{2}(t_1^2+2\rho t_1t_2+t_2^2)},\ p = \mathrm{E}\left(\frac{(\xi^2-1)\xi}{\sqrt{\alpha_4-1}}\right) = \frac{\alpha_3}{\sqrt{\alpha_4-1}}$$

and $\psi(it_1, it_2)$ is given in Lemma 3 by taking therein $\zeta_1 = \dfrac{\xi^2 - 1}{\sqrt{\alpha_4 - 1}}$, $\zeta_2 = \xi, \xi$ being any of the ξ_i's.

We write

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(x,y-u)\,dW \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (G(x,y-u) - H(x,y-u))\,dW \\
&- \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (G(x,y-u) - H(x,y-u))(w(x,y) + \gamma(x,y))\,dy\,dx \\
&+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(x,y-u)\,dW \\
&- \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(x,y-u)(w(x,y) + \gamma(x,y))\,dy\,dx \\
&+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(x,y-u)(w(x,y) + \gamma(x,y))\,dy\,dx
\end{aligned} \tag{6}
$$

5. We have

$$
|G(x, y-u) - H(x, y-u)| \leqq 1 - e^{-\epsilon x^{2l}} \leqq \epsilon x^{2l} \tag{7}
$$

$$
\left|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (G(x,y-u) - H(x,y-u))\,dW\right| \leqq \int_{-\infty}^{\infty} \epsilon x^{2l}\,dW = \epsilon E(X^{2l}) \leqq Q_k\epsilon \tag{8}
$$

since

$$
E(X^{2l}) \leqq A_k E\left(\frac{\xi^2 - 1}{\sqrt{\alpha_4 - 1}}\right)^{2l} \leqq Q_k
$$

where Q_k depends on $\alpha_3, \cdots, \alpha_{2k}$.

Similarly,

$$
\left|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (G(x,y-u) - H(x,y-u))(w(x,y) + \gamma(x,y))\,dy\,dx\right| \leqq Q_k\epsilon \tag{9}
$$

Next,

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(x,y-u)(w(x,y) + \gamma(x,y))\,dy\,dx \\
&= \iint\limits_{y \leqq u+b+L(x)} (w(x,y) + \gamma(x,y))\,dy\,dx - \iint\limits_{y \leqq u+b} (w(x,y) + \gamma(x,y))\,dy\,dx
\end{aligned} \tag{10}
$$

where the first term on the right-hand side, regarded as a function of $n^{-\frac{1}{2}}$, has a Taylor expansion in powers of $n^{-\frac{1}{2}}$, whose first few terms we shall compute in

section 9; for the present let us denote it by $B(u+b)+C(u+b)n^{-\frac{1}{2}(k-2)}$ where $C=C(u+b)$ is a constant depending on k, $P(x)$ and z, a more explicit estimate of which will be given in section 10.

Further, we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(x,y-u)\,dW = \int\int_{y\leqq u+b+L(x)} dW - \int\int_{y\leqq u+b} dW \tag{11}$$

by Cramér's asymptotic expansion for the mean $\sqrt{n}Y$, and as is also shown in Hsu's paper we have

$$\int\int_{y\leqq u+b} dW - \int\int_{y\leqq u+b} (w(x,y)+\gamma(x,y))\,dy\,dx = \Lambda_k n^{-\frac{1}{2}(k-2)} \tag{12}$$

Collecting all the results from (5)–(12), we get

$$\begin{aligned}
&\int\int_{y\leqq u+b+L(x)} dW - B(u+b) - C(u+b)n^{-\frac{1}{2}(k-2)} \\
&\quad = \Lambda_k(\epsilon + n^{-\frac{1}{2}(k-2)}) + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(x,y-u)\,dW \\
&\quad - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(x,y-u)(w(x,y)+\gamma(x,y))\,dy\,dx
\end{aligned}$$

Now we use A. C. Berry's weighting factor $\dfrac{1-\cos Tu}{u^2}$ and obtain

$$\begin{aligned}
&\int_{-\infty}^{\infty} \frac{1-\cos Tu}{u^2}\left(\int\int_{y\leqq u+b+L(x)} dW - B(u+b) - C(u+b)\right)du \\
&\qquad = \Lambda_k T(\epsilon + n^{-\frac{1}{2}(k-2)}) \\
&\qquad + \int_{-\infty}^{\infty} \frac{1-\cos Tu}{u^2}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(x,y-u)\,dW\right. \\
&\qquad \left. - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(x,y-u)(w(x,y)+\gamma(x,y))\,dy\,dx\right)du
\end{aligned} \tag{13}$$

since

$$\int_{-\infty}^{\infty} \frac{1-\cos Tu}{u^2}\,du = \pi T\,.$$

6. To transform the triple integral on the right-hand side of (13) we use the Fourier transform as Hsu did.

Let

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-it_1x-it_2y}H(x,y)\,dy\,dx = h(t_1,t_2)\,;$$

then

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-it_1x-it_2y}H(x,y-u)\,dy\,dx = e^{-it_2u}h(t_1,t_2)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-it_1x-it_2y}\left(\int_{-\infty}^{\infty}\frac{1-\cos Tu}{u^2}H(x,y-u)\,du\right)dy\,dx$$

$$= \begin{cases}\pi(T-|t_2|)h(t_1,t_2) \text{ if } |t_2|\le T\\ 0 \qquad\qquad \text{otherwise}\end{cases}$$

since

$$\int_{-\infty}^{\infty}\frac{1-\cos Tu}{u^2}e^{-it_2u}\,du = \begin{cases}\pi(T-|t_2|) & \text{if } |t_2|\le T\\ 0, & \text{otherwise}\end{cases}$$

By Fourier inversion we have, almost everywhere,

$$\int_{-\infty}^{\infty}\frac{1-\cos Tu}{u^2}H(x,y-u)\,du = \frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-T}^{T}e^{it_1x+it_2y}(T-|t_2|)h(t_1,t_2)\,dt_2\,dt_1$$

Hence

$$\int_{-\infty}^{\infty}\frac{1-\cos Tu}{u^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}H(x,y-u)\,dW\,du$$

$$= \frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-T}^{T}(T-|t_2|)f(t_1,t_2)h(t_1,t_2)\,dt_2\,dt_1 \tag{14}$$

Similarly we obtain

$$\int_{-\infty}^{\infty}\frac{1-\cos Tu}{u^2}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}H(x,y-u)(w(x,y)+\gamma(x,y))\,dy\,dx\right)du$$

$$= \frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-T}^{T}(T-|t_2|)\phi(t_1,t_2)\{1+\psi(it_1,it_2)\}h(t_1,t_2)\,dt_2\,dt_1 \tag{15}$$

From (14) and (15) we obtain

$$\int_{-\infty}^{\infty}\frac{1-\cos Tu}{u^2}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}H(x,y-u)\,dW - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}H(x,y-u)(w(x,y)+\gamma(x,y))\,dy\,dx\right)du$$

$$= \frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-T}^{T}(T-|t_2|)\{f(t_1,t_2) - \phi(t_1,t_2)(1+\psi(it_1,it_2))\}h(t_1,t_2)\,dt_2\,dt_1. \tag{16}$$

7. To estimate the double integral on the right-hand side of (16) we break it up into parts and use the following estimates of $h(t_1,t_2)$.

LEMMA B. *We have for* $l = \left[\frac{k}{2}\right] \ge 1$,

$$(1) \qquad |h(t_1, t_2)| \leqq A_k |z| \sum_{j=1}^{2l} (\alpha_4 - 1)^{\frac{1}{4}j} n^{-\frac{1}{2}j} e^{-(j+1)/2l};$$

$$(2) \qquad |h(t_1, t_2)| \leqq Q_k l^{-2} z^2 N(|t_2|, n^{-1/2}, \epsilon^{-1/2l})$$

where $N(|t_2|, n^{-1/2}, \epsilon^{-1/2l})$ *is a polynomial with constant coefficients in the indicated arguments.*

PROOF.

$$|h(t_1, t_2)| = \iint_{b<y\leqq b+L(x)} e^{-it_1x - it_2y - \epsilon x^{2l}}\, dy\, dx - \iint_{b+L(x)<y\leqq b} e^{-it_1x - it_2y - \epsilon x^{2l}}\, dy\, dx$$

$$= \left(\int_{L(x)\geqq 0}\int_b^{b+L(x)} - \int_{L(x)<0}\int_{b+L(x)}^{b}\right) e^{-it_1x - it_2y - \epsilon x^{2l}}\, dy\, dx$$

$$= (-it_2)^{-1} e^{-it_2b} \int_{-\infty}^{\infty} e^{-it_1x - \epsilon x^{2l}} (e^{it_2L(x)} - 1)\, dx.$$

Hence

$$|h(t_1, t_2)| \leqq |t_2|^{-1} \int_{-\infty}^{\infty} |t_2 L(x)|\, e^{-\epsilon x^{2l}}\, dx.$$

Since

$$|L(x)| \leqq A_k |z| \sum_{j=1}^{2l} (\alpha_4 - 1)^{\frac{1}{4}j} n^{-\frac{1}{2}j} |x|^j$$

we obtain

$$|h(t_1, t_2)| \leqq A_k |z| \sum_{j=1}^{2l} (\alpha_4 - 1)^{\frac{1}{4}j} n^{-\frac{1}{2}j} \int_{-\infty}^{\infty} |x|^j e^{-\epsilon x^{2l}}\, dx$$

$$\leqq A_k |z| \sum_{j=1}^{2l} (\alpha_4 - 1)^{\frac{1}{4}j} n^{-\frac{1}{2}j} \epsilon^{-(j+1)/2l}.$$

Next, we write

$$h(t_1, t_2) = (-it_2)^{-1} e^{-it_2b} \int_{-\infty}^{\infty} u''(x) v(x)\, dx$$

with

$$u''(x) = e^{-it_1x}, \qquad v(x) = e^{-\epsilon x^{2l}} (e^{-it_2L(x)} - 1).$$

Integrating by parts twice, we get

$$h(t_1, t_2) = (-it_2)^{-1} e^{-it_2b} \int_{-\infty}^{\infty} u(x) v''(x)\, dx$$

whence

$$|h(t_1, t_2)| \leqq A_k t_1^{-2} \int_{-\infty}^{\infty} e^{-\epsilon x^{2l}} \{|L''(x)| + \epsilon x^{2l-1} |L'(x)| + \epsilon |x|^{2l-2} |L(x)| + \epsilon^2 x^{2l-2} |L(x)| + |t_2| |L'^2(x)|\} dx \leqq Q_k t_1^{-2} (|z| + z^2) N(|t_2|, n^{-\frac{1}{2}}, \epsilon^{-\frac{1}{2l}})$$

The lemma is proved.

Now we write

$$\int_{-\infty}^{\infty} \int_{-T}^{T} (T - |t_2|)\{f - \phi(1 + \psi)\} h \, dt_2 \, dt_1$$

$$= \iint_{\substack{|t_1| \leqq Q_k n^{\frac{1}{2}} \\ |t_2| \leqq Q_k n^{\frac{1}{2}}}} + \iint_{\substack{|t_1| > Q_k n^{\frac{1}{2}} \\ |t_2| \leqq T}} + \iint_{\substack{|t_1| \leqq Q_k n^{\frac{1}{2}} \\ Q_k n^{\frac{1}{2}} < |t_2| \leqq T}} = I_1 + I_2 + I_3 . \tag{17}$$

On I_1 we use Lemma 3 and Lemma B, (1):

$$|I_1| \leqq Q_k |z| \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} T \left(\sum_{j=1}^{l} n^{-\frac{1}{2}j} \epsilon^{-\frac{1}{2}(k-2)} \right) n^{-\frac{1}{2}(k-2)} \cdot \left\{ \sum_{i=1}^{2} (|t_i|^k + \cdots + |t_i|^{3(k-2)} \right\} e^{-\frac{1}{2}(1-\rho^2)(t_1^2 + t_2^2)} \, dt_2 \, dt_1$$

$$\leqq Q_k |z| \, T n^{-\frac{1}{2}(k-2)} \sum_{j=1}^{2l} n^{-\frac{1}{2}j} \epsilon^{-(j+1)/2l} .$$

On I_2 we use Lemma 7 and Lemma B (2). Since $|t_1| > Q_k n^{\frac{1}{2}}$, $|\Phi(1 + \psi)| \leqq e^{-nQ_k}$, and by Lemma 7, $p(t_1 n^{-\frac{1}{2}}, t_2 n^{-\frac{1}{2}}) = e^{-Q_k}$ so that $|f(t_1, t_2)| \leqq e^{-nQ_k}$, $|f - \Phi(1 + \psi)| \leqq e^{-nQ_k}$

$$I_2 \leqq Q_k z^2 \iint_{\substack{|t_1| > Q_k n^{\frac{1}{2}} \\ |t_2| \leqq T}} T t_1^{-2} e^{-nQ_k} N(|t_2|, n^{-\frac{1}{2}}, \epsilon^{-1/2l}) \, dt_2 \, dt_1 .$$

Let $\epsilon = n^{-\beta}$, $\beta > 0$, then it is evident that

$$|I_2| \leqq Q_k z^2 .$$

Similarly using Lemma 7 and Lemma B, (1) on I_3 we see that

$$|I_3| \leqq Q_k |z| .$$

Therefore

$$\left| \frac{1}{4\pi} \int_{-\infty}^{\infty} \int_{-T}^{T} (T - |t_2|)\{f(t_1, t_2) - \phi(t_1, t_2)(1 + \psi(it_1, it_2))\} h(t_1, t_2) \, dt_2 \, dt_1 \right| \leqq Q_k \left(|z| + z^2 + |z| \, T n^{-\frac{1}{2}(k-2)} \sum_{j=1}^{2l} n^{-\frac{1}{2}j} \epsilon^{-(j+1)/2l} \right). \tag{18}$$

8. Combining (13), (16), (17) we obtain

$$\left|\int_{-\infty}^{\infty}\frac{1-\cos Tu}{u^2}\left(\iint\limits_{y\leqq u+b+L(x)} dW - B(u+b) - C(u+b)n^{-\frac{1}{2}(k-2)}\right)du\right| \tag{19}$$

$$\leqq Q_k\left(T\epsilon + Tn^{-\frac{1}{2}(k-2)} + |z| + z^2 + |z|\,Tn^{-\frac{1}{2}(k-2)}\sum_{j=1}^{2l} n^{-\frac{1}{2}j}\,\epsilon^{-(j+1)/2l}\right).$$

Now we shall choose T and ϵ suitably. Let

$$T = n^{\alpha}, \qquad \epsilon = n^{-\beta}, \qquad \alpha > 0, \qquad \beta > 0.$$

To make the right-hand side of (19) a constant depending on z only, we must have $\alpha \leqq \frac{1}{2}(k-2)\cdot\beta \leqq \alpha$. Then

$$\sum_{j=1}^{2l} n^{-\frac{1}{2}j}\,\epsilon^{-(j+1)/2l} = \sum_{j=1}^{2l} n^{((\beta-1)j+\beta)/2l}.$$

We must choose $\beta < k/2$, then

$$\sum_{j=1}^{2l} n^{-\frac{1}{2}j}\,\epsilon^{-(j+1)/2l} \leqq A_k n^{(2\beta-1)/2l}.$$

To make the exponent as small as possible we choose $\beta = \alpha$, then

$$|z|\,Tn^{-\frac{1}{2}(k-2)}\sum_{j=1}^{2l} n^{-\frac{1}{2}j}\,\epsilon^{-(j+1)/2l} \leqq A_k|z|\,n^{\alpha-\frac{1}{2}(k-2)+(2\alpha-1)/2l} = A_k|z|\,n^{(l+1)\alpha/l-\frac{1}{2}(k-1)}$$

since α is to be as large as possible, we choose

$$\alpha = \alpha_0 = \min\left(\frac{k-2}{2}, \frac{(k-1)l}{2(l+1)}\right), \qquad l = \left[\frac{k}{2}\right].$$

Then we obtain

$$\left|\int_{-\infty}^{\infty}\frac{1-\cos Tu}{u^2}\left(\iint\limits_{y\leqq u+b+L(x)} dW - B(u+b) - C(u+b)n^{-\frac{1}{2}(k-2)}\right)du\right| \tag{20}$$

$$\leqq Q_k(1+z^2).$$

Let $F^*(u)$ be the distribution function of $Y - L(X)$, and let

$$F_1(u) = B(u) + C(u)n^{-\frac{1}{2}(k-2)}$$

Then we may write (20) as

$$\left|\int_{-\infty}^{\infty}\frac{1-\cos Tu}{u^2}\{F^*(u+b) - F_1(u+b)\}\,du\right| \leqq Q_k(1+z^2). \tag{21}$$

By the definition of $F_1(u)$ we see that the conditions in Lemma 8 are all satisfied with a certain constant D depending on k, $P(x)$, and z for the M therein. Then choosing b to be the a in Lemma 8, we obtain from Lemma 8 and (21),

$$DT\delta\left\{3\int_0^{T\delta}\frac{1-\cos x}{x^2}\,dx - \pi\right\} \leqq Q_k(1+z^2) \tag{22}$$

where

$$\delta = \frac{1}{2D} \text{ l.u.b. } | F^*(u) - F_1(u) |.$$

Now there exists A such that if $T\delta > A$, then

$$3\int_0^{T\delta} \frac{1 - \cos x}{x^2}\, dx - \pi > 1,$$

hence it follows from (22) that

$$T\delta \leqq \max (A, D^{-1}Q_k(1 + z^2)).$$

Thus for another Q_k' exceeding both A and the above Q_k, we have

$$T\delta \leqq Q_k'(1 + z^2)$$

and so finally, dropping the prime,

$$| F^*(u) - F_1(u) | \leqq Q_k(1 + z^2)DT^{-1} = Q_k(1 + z^2)Dn^{-\alpha_0}. \tag{23}$$

In particular, taking b to be $z(1 + n^{-1}z^2)^{-\frac{1}{2}} = z'$, say:

$$Pr\{Y - L(X) \leqq z'\} = B(z') + C(z')n^{-\frac{1}{2}(k-2)} + \Lambda_k(1 + z^2)Dn^{-\alpha} \tag{24}$$

where

$B(z') + C(z')n^{-\frac{1}{2}(k-2)}$ = the Taylor expansion with a remainder of

$$\iint\limits_{y - L(x) \leqq z'} (w(x, y) + \gamma(x, y))\, dy\, dx.$$

and D is an upper bound for

$$| B'(u) + C'(u)n^{-\frac{1}{2}(k-2)} |.$$

9. Let $\lambda = n^{-\frac{1}{2}}$, and rewrite the $z' + L_{2l-1}(x)$, $l \geqq 2$ there as $g(\lambda)$:

$$g(\lambda) = z'\left(1 + \frac{(\alpha_4 - 1)^{1/2}}{2}\lambda x - \frac{\alpha_4 - 1}{8}\lambda^2 x^2 + \frac{(\alpha_4 - 1)^{3/2}}{16}\lambda^3 x^3 + \cdots\right)$$

Then

$$g(0) = z'$$

$$g'(0) = \frac{(\alpha_4 - 1)^{\frac{1}{2}}}{2} z'x$$

$$g''(0) = -\frac{\alpha_4 - 1}{4} z'x^2$$

$$g'''(0) = \frac{3(\alpha_4 - 1)^{3/2}}{8} z'x^3.$$

Let $p \geqq 0$, $q \geqq 0$; $w_{pq}(x, y) = \dfrac{\partial^{p+q}}{\partial x^p \partial y^q} w(x, y)$ where $w(x, y)$ is defined in section 4 and we know that

$$w(x, y) = \frac{1}{2\pi\sqrt{1-\rho^2}}\, e^{-(x^2-2\rho xy+y^2)/2(1-\rho^2)}.$$

Let

$$f_{pq}(\lambda) = \int_{-\infty}^{\infty}\int_{-\infty}^{g(\lambda)} w_{pq}(x, y)\, dy\, dx. \tag{26}$$

Then

$$\begin{aligned}
f'_{pq}(\lambda) &= \int_{-\infty}^{\infty} g'(\lambda) w_{pq}(x, g(\lambda))\, dx \\
f''_{pq}(\lambda) &= \int_{-\infty}^{\infty} (g''(\lambda) w_{pq}(x, g(\lambda)) + g'^2(\lambda) w_{p,q+1}(x, g(\lambda)))\, dx \qquad (27) \\
f'''_{pq}(\lambda) &= \int_{-\infty}^{\infty} (g'''(\lambda) w_{pq}(x, g(\lambda)) + 3g''(\lambda)g'(\lambda) w_{p,q+1}(x, g(\lambda)) \\
&\qquad + g'^3(\lambda) w_{p,q+2}(x, g(\lambda)))\, dx
\end{aligned}$$

Let

$$\Phi = \Phi(z') = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z'} e^{\frac{1}{2}y^2}\, dy, \quad \Phi^{(q)} = \frac{d^q}{dz^q}\Phi(z)\Big|_{z=z'}$$

$$I^r_{pq} = \int_{-\infty}^{\infty} x^r w_{pq}(x, z')\, dx \tag{28}$$

We have computed the following table of values of I^r_{pq}:

r \ p	0	1	2	3	$\geqq 4$
0	$\Phi^{(q)}$	0	0	0	0
1	$-\rho\Phi^{(q+1)}$	$-\Phi^{(q)}$	0	0	0
2	$\Phi^{(q)} + \rho^2\Phi^{(q+2)}$	$2\rho\Phi^{(q+1)}$	$2\Phi^{(q)}$	0	0
3	$-3\rho\Phi^{(q+1)} - \rho^3\Phi^{(q+3)}$	$-3\Phi^{(q)} - 3\rho^2\Phi^{(q+2)}$	$-6\rho\Phi^{(q+1)}$	$-6\Phi^{(q)}$	0

Next, we find, from (25)–(28),

$$\begin{aligned}
f_{00}(0) &= \Phi; \\
f_{pq}(0) &= I^0_{p,q-1} \text{ for } q \geqq 1; \\
f'_{pq}(0) &= \frac{(\alpha_4 - 1)^{\frac{1}{2}}}{4} z' I^1_{pq} \qquad (92) \\
f''_{pq}(0) &= -\frac{\alpha_4 - 1}{4} z' I^2_{pq} + \frac{\alpha_4 - 1}{4} z'^2 I^2_{p,q+1} \\
f'''_{pq}(0) &= \frac{3(\alpha_4 - 1)^{3/2}}{8} z' I^3_{pq} - \frac{3(\alpha_4 - 1)^{3/2}}{8} z'^2 I^3_{p,q+1} + \frac{(\alpha_4 - 1)^{3/2}}{8} z'^3 I^3_{p,q+2}
\end{aligned}$$

Now we can expand

$$\iint\limits_{y\leqq b+L(x)} w(x,y)\,dy\,dx = \int_{-\infty}^{\infty}\int_{-\infty}^{g(\lambda)} w(x,y)\,dy\,dx = f_{00}(\lambda)$$

Write the Taylor's series for $f_{00}(\lambda)$:

$$f_{00}(\lambda) = f_{00}(0) + f'_{00}(0)\lambda + \frac{f''_{00}(0)}{2}\lambda^2 + \frac{f'''_{00}(0)}{6}\lambda^3 + \cdots$$

Substituting from (29), we get

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{g(\lambda)} w(x,y)\,dy\,dx &= \Phi - \frac{(\alpha_4-1)^{1/2}}{2n^{1/2}}\rho z'\Phi^{(1)} \\ &+ \frac{\alpha_4-1}{8n}\{-z'(\Phi^{(0)}+\rho^2\Phi^{(2)}) + z'^2(\Phi^{(1)}+\rho^2\Phi^{(3)})\} \\ &+ \frac{(\alpha_4-1)^{3/2}}{48n^{3/2}}\{3z'(-3\rho\Phi^{(1)}-\rho^3\Phi^{(3)}) - 3z'^2(-3\rho\Phi^{(2)}-\rho^3\Phi^{(4)}) \\ &\qquad + z'^3(-3\rho\Phi^{(3)}-\rho^3\Phi^{(5)})\} + \cdots\end{aligned} \tag{30}$$

Further, we must obtain the beginning terms of $\gamma(x, y)$ as given in Lemma 3, for which purpose we refer to Hsu's paper. We have, in fact

$$\psi(it_1,it_2) = -\frac{iU^3}{6n^{1/2}} + \left(\frac{U_4}{24} - \frac{U_3^2}{36}\right)\frac{1}{n} + \left(\frac{U_5}{120} - \frac{U_3U_4}{72} + \frac{U_3^2}{216}\right)\frac{i}{n^{3/2}} + \cdots$$

where

$$\begin{aligned}U_3 &= E\left(t_1\frac{\xi^2-1}{\sqrt{\alpha_4-1}} + t_2\xi\right)^3 \\ &= \alpha_3 t_2^3 + 3\sqrt{\alpha_4-1}\,t_2^2t_1 + 3\frac{\alpha_5-2\alpha_3}{\alpha_4-1}t_2t_1^2 + \frac{\alpha_6-3\alpha_4+2}{(\alpha_4-1)^{3/2}}t_1^3 \\ U_4 &= E\left(t_1\frac{\xi^2-1}{\sqrt{\alpha_4-1}} + t_2\xi\right)^4 - 3\left(E\left(t_1\frac{\xi^2-1}{\sqrt{\alpha_4-1}} + t_2\xi\right)\right)^2 \\ &= (\alpha_4-3)t_2^4 + 4\frac{\alpha_5-4\alpha_3}{\sqrt{\alpha_4-1}}t_2^3t_1 + \cdots \\ U_5 &= E\left((t_1\frac{\xi^2-1}{\sqrt{\alpha_4-1}} + t_2\xi\right)^5 - 10E\left(t_1\frac{\xi^2-1}{\sqrt{\alpha_4-1}} + t_2\xi\right)^2 E\left(t_1\frac{\xi^2-1}{\sqrt{\alpha_4-1}} + t_2\xi\right)^3 \\ &= (\alpha_5-10\alpha_3)t_2^5 + \cdots\end{aligned}$$

To avoid the exhibition of very long expressions, let us separate the terms in $\psi(it_1, it_2)$ according to the powers of $n^{-1/2}$, and denote the terms of the power $n^{-1/2}\,n^{-1}, n^{-3/2}$ by ψ_1, ψ_2, ψ_3, respectively.

Thus $\psi_1 = -iU_3/6n^{1/2}$, and the corresponding $\gamma(x, y)$ is

$$\gamma_1(x,y) = -\frac{1}{6n^{1/2}}\left(\alpha_3 w_{03}(x,y) + 3\sqrt{\alpha_4 - 1}\, w_{12}(x,y) + 3\frac{\alpha_5 - 2\alpha_3}{\alpha_4 - 1} w_{21}(x,y) + \cdots\right) \tag{31}$$

where, as hereafter, the terms omitted will yield nothing in the long run.

Now we have by (31) and (26),

$$\int_{-\infty}^{\infty}\int_{-\infty}^{g(\lambda)} \gamma_1(x, y)\, dy\, dx$$

$$= -\frac{1}{6n^{1/2}}\left(\alpha_3 f_{03}(\lambda) + 3\sqrt{\alpha_4 - 1} f_{12}(x, y) + 3\frac{\alpha_5 - 2\alpha_3}{\alpha_4 - 1} f_{21}(x, y) + \cdots\right)$$

$$= -\frac{1}{6n^{1/2}}\left(\alpha_3 f_{03}(0) + 3\sqrt{\alpha_4 - 1} f_{12}(0) + 3\frac{\alpha_5 - 2\alpha_3}{\alpha_4 - 1} f_{21}(0) + \cdots\right)$$

$$-\frac{1}{6n}\left(\alpha_3 f'_{03}(0) + 3\sqrt{\alpha_4 - 1} f'_{12}(0) + 3\frac{\alpha_5 - 2\alpha_3}{\alpha_4 - 1} f'_{21}(0) + \cdots\right)$$

$$-\frac{1}{12n^{3/2}}\left(\alpha_3 f''_{03}(0) + 3\sqrt{\alpha_4 - 1} f''_{12}(0) + 3\frac{\alpha_5 - 2\alpha_3}{\alpha_4 - 1} f''_{21}(0) + \cdots\right)$$

$$= -\frac{1}{6n^{1/2}}(\alpha_3 l^0_{02}) - \frac{(\alpha_4 - 1)^{1/2}}{12n} z'(\alpha_3 l^1_{03} + 3\sqrt{\alpha_4 - 1}\, l^1_{12}) \tag{32}$$

$$+\frac{\alpha_4 - 1}{48n^{3/2}}\left\{z'\left(\alpha_3 l^2_{03} + 3\sqrt{\alpha_4 - 1}\, l^2_{12} + 3\frac{\alpha_5 - 2\alpha_3}{\alpha_4 - 1} l^2_{21}\right)\right.$$

$$\left. - z'^2\left(\alpha_3 l^2_{04} + 3\sqrt{\alpha_4 - 1}\, l^2_{13} + 3\frac{\alpha_5 - 2\alpha_3}{\alpha_4 - 1} l^2_{22}\right)\right\} + \cdots$$

$$= -\frac{\alpha_3}{6n^{1/2}}\Phi^{(2)} + \frac{(\alpha_4 - 1)^{1/2}}{12n} z'(\alpha_3 \rho\Phi^{(4)} + 3\sqrt{\alpha_4 - 1}\,\Phi^{(2)})$$

$$+\frac{\alpha_4 - 1}{48n^{3/2}}\left\{z'\left[\alpha_3(\Phi^{(3)} + \rho^2\Phi^{(5)}) + 6\sqrt{\alpha_4 - 1}\,\rho\Phi^{(3)} + 6\frac{\alpha_5 - 2\alpha_3}{\alpha_4 - 1}\Phi^{(1)}\right]\right.$$

$$\left. - z'^2\left[\alpha_3(\Phi^{(4)} + \rho^2\Phi^{(6)}) + 6\sqrt{\alpha_4 - 1}\,\rho\Phi^{(4)} + 6\frac{\alpha_5 - 2\alpha_3}{\alpha_4 - 1}\Phi^{(2)}\right]\right\} + \cdots.$$

Similarly, omitting the intermediate steps to save space, we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{g(\lambda)} \gamma_2(x, y)\, dy\, dx = \frac{1}{72n}\{3(\alpha_4 - 3)\Phi^{(3)} + 2\alpha_3^2\Phi^{(5)}\}$$

$$-\frac{(\alpha_4 - 1)^{1/2}}{144n^{3/2}} z'\left\{3(\alpha_4 - 3)\rho\Phi^{(5)} + 12\frac{\alpha_5 - 4\alpha_3}{\sqrt{\alpha_4 - 1}}\Phi^{(3)} + 2\alpha_3^2\rho\Phi^{(7)} + 12\alpha_3\sqrt{\alpha_4 - 1}\,\Phi^{(5)}\right\} + \cdots. \tag{33}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{g(\lambda)} \gamma_3(x, y)\, dy\, dx$$

$$= -\frac{1}{n^{3/2}}\left(\frac{\alpha_5 - 10\alpha_3}{120}\Phi^{(4)} + \frac{\alpha_3\alpha_4}{72}\Phi^{(6)} + \frac{\alpha_1^3}{216}\Phi^{(8)}\right) + \cdots. \tag{34}$$

Combining (30), (32)–(34) and simplifying, we obtain, as the first four terms of the asymptotic expansion of $F(z)$:

$$\begin{aligned}
\iint_{y-L(x)\leqq z'} (w(x, y) + \gamma(x, y))\, dy\, dx &= \Phi - \frac{\alpha_3}{6n^{1/2}}(3z'\Phi^{(1)} + \Phi^{(2)}) \\
&+ \frac{1}{4n}\left\{\frac{\alpha_4 - 3}{6}\Phi^{(3)} + \frac{\alpha_3^2}{9}\Phi^{(9)}\right. \\
&+ z'\left(\frac{\alpha_3^2}{3}\Phi^{(4)} + \frac{2(\alpha_4 - 1) - \alpha_3^2}{2}\Phi^{(2)} - \frac{\alpha_4 - 1}{2}\Phi^{(0)}\right) \\
&\left.+ z'^2\left(\frac{\alpha_4 - 1}{2}\Phi^{1)} + \frac{\alpha_3^2}{2}\Phi^{(3)}\right)\right\} \\
&+ \frac{1}{24n^{3/2}}\left\{-\frac{\alpha_5 - 10\alpha_3}{5}\Phi^{(4)} + \frac{\alpha_3\alpha_4}{3}\Phi^{(6)} + \frac{\alpha_3^3}{9}\Phi^{(8)}\right. \\
&+ z'\left[\frac{6\alpha_5 - 3\alpha_3 - 9\alpha_3\alpha_4}{2}\Phi^{(1)} + \frac{7\alpha_3(\alpha_4 - 1 - \alpha_3^2) - 2\alpha_5}{2}\Phi^{(3)}\right. \\
&\left.+ \frac{\alpha_3(\alpha_3^2 - 5\alpha_4 + 7)}{2}\Phi^{(5)} - \frac{\alpha_3^3}{3}\Phi^{(7)}\right] \\
&+ z'^2\left[\frac{9\alpha_3\alpha_4 + 3\alpha_3 - 6\alpha_5}{2}\Phi^{(2)} + \frac{\alpha_3(3\alpha_3^2 - 7\alpha_4 + 7)}{2}\Phi^{(4)}\right] \\
&\left.+ z'^3\left[\frac{-3\alpha_3(\alpha_4 - 1)}{2}\Phi^{(3)} - \frac{\alpha_3^2}{2}\Phi^{(5)}\right]\right\} + \cdots.
\end{aligned} \tag{35}$$

10. In order to estimate the remainder $C(z')n^{-(k-2)/2}$ in the Taylor expansion we write, in accordance with Lemma 3,

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{g(\lambda)} (w(x, y) + \gamma(x, y))\, dy\, dx \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{g(\lambda)} \left\{w(x, y) + \sum_{\nu=1}^{k-3} \lambda^\nu \Sigma(-1)^{\nu_1+\nu_2} a_{\nu_1\nu_2} w_{\nu_1\nu_2}(x, y)\right\} dy\, dx \\
&= f_{00}(\lambda) + \sum_{\nu=1}^{k-3} \lambda^\nu \Sigma(-1)^{\nu_1+\nu_2} a_{\nu_1\nu_2} f_{\nu_1\nu_2}(\lambda) = \sum_{j=0}^{k-3} f_{00}^{(j)}(0)\frac{\lambda^j}{j!} \\
&\quad + f_{00}^{(k-2)}(\theta\lambda)\frac{\lambda^{k-2}}{(k-2)!} + \sum_{\nu=1}^{k-3} \lambda^\nu \Sigma(-1)^{\nu_1+\nu_2} a_{\nu_1\nu_2} \\
&\quad \cdot \left\{\sum_{j=1}^{k-3-\nu} f_{\nu_1\nu_2}^{(j)}(0)\frac{\lambda^j}{j!} + f_{\nu_1\nu_2}^{(k-2-\nu)}(\theta\lambda)\frac{\lambda^{k-2-\nu}}{(k-2-\nu)!}\right\}
\end{aligned}$$

$$= B(z') + f_{00}^{(k-2)}(\theta\lambda)\frac{\lambda^{k-2}}{(k-2)!} + \sum_{\nu=1}^{k-3}\Sigma(-1)^{\nu_1+\nu_2}a_{\nu_1\nu_2}f_{\nu_1\nu_2}^{(k-2-\nu)}(\theta\lambda)\frac{\lambda^{k-2}}{(k-2-\nu)!}$$

$$= B(z') + \Lambda_k\lambda^{k-2}\left(f_{00}^{(k-2)}(\theta\lambda) + \sum_{\nu=1}^{k-3}f_{\nu_1\nu_2}^{(k-2-\nu)}(\theta\lambda)\right).$$

Thus

$$C(z') = \Lambda_k\left(f_{00}^{(k-2)}(\theta\lambda) + \sum_{\nu=1}^{k-3}f_{\nu_1\nu_2}^{(k-2-\nu)}(\theta\lambda)\right)$$

Now we may write

$$f_{\nu_1\nu_2}^{(k-2-\nu)}(\theta\lambda) = \int_{-\infty}^{\infty}\Sigma(\Pi(g^{(s)}(\theta\lambda))^t)w_{pq}(x, g(\theta\lambda))\,dx$$

where, if we attach a weight s to $g^{(s)}(\theta\lambda)$, the polynomial under the integral sign is isobaric of weight $k-2-\nu$ in these $g^{(s)}$'s, and the coefficient of each term is a constant multiple of a certain $w_{pq}(x, g(\theta\lambda))$. Further, it is easily seen by induction that we have

$$g^{(s)}(\theta\lambda) = P_{1+2s}(z)(1+\theta^2\lambda^2z^2)^{-\frac{1}{2}-s}$$

where $P_{1+2s}(z)$ is a polynomial of the three variables z, x, $\theta\lambda$ which is of at most the $(1+2s)$th degree in z and of the $(2l-1)$st degree in x, and whose coefficients are all Λ_k.

Therefore,

$$\begin{aligned}|f_{\nu_1\nu_2}^{(k-2-\nu)}(\theta\lambda)| &\leqq \int_{-\infty}^{\infty}Q_k(|x| + x^2 + \cdots + |x|^{2l-1})\\ &\quad\cdot(1+|z|^{(1+2s)(k-2-\nu)s^{-1}})w_{pq}(x, g(\theta\lambda))\,dx\\ &\leqq \int_{-\infty}^{\infty}Q_k(|x| + x^2 + \cdots + |x|^{2l-1})(1+|z|^{1+2(k-2)})e^{-\frac{1}{2}x^2}\,dx\\ &\leqq Q_k(1+|z|^{2k-3})\end{aligned}$$

Thus

$$C(z') \leqq Q_k(1+|z|^{2k-3}) \tag{36}$$

Lastly, an estimate of D is easy:

$$\begin{aligned}|F_1'(u)| &= \left|\frac{d}{du}\int_{-\infty}^{\infty}\int_{-\infty}^{u+L(x)}(w(x, y) + \gamma(x, y))\,dy\,dx\right|\\ &\leqq \int_{-\infty}^{x}(w(x, u+L(x)) + |\gamma(x, u+L(x))|)\,dx \leqq Q_k.\end{aligned} \tag{37}$$

Collecting the results of (24), (36), (37) we obtain

$$\Pr\{Y - L(X) \leqq z'\} = B(z') + \Lambda_k((1+|z|^{2k-3})n^{-\frac{1}{2}(k-2)} + (1+z^2)n^{-\alpha_0}),$$

$$\alpha_0 = \min\left(\frac{k-2}{2}, \quad \frac{(k-1)\left[\frac{k}{2}\right]}{2\left(\left[\frac{k}{2}\right]+1\right)}\right). \tag{38}$$

Or, more simply,

$$|\, Pr\{Y - L(X) \leqq z'\} - B(z')\,| \leqq Q_k(1 + |\, z\,|^{2k-3})n^{-\alpha_0}, \tag{39}$$

where the first four terms of $B(z')$ are given by (35).

12. To return to $F(z)$. We see that $B(z')$ depends on the function $L(x)$. Recalling section 3 we now write B_m for the B corresponding to L_m, with $m = 2l - 1$ or $2l$.

Then by (4) the value of $F(z)$ lies between

$$\Pr\{Y - L_{2l-1}(X) \leqq z'\} \quad \text{and} \quad \Pr\{Y - L_{2l}(X) \leqq z'\}.$$

From the asymptotic expansion just obtained for either of them, we see that the absolute value of their difference does not exceed

$$|\, B_{2l-1}(z') - B_{2l}(z')\,| + Q_k(1 + |\, z\,|^{2k-3})n^{-\alpha_0}.$$

But

$$L_{2l}(x) = L_{2l-1}(x) - z' b_{2l}(\alpha_4 - 1)^l n^{-l} x^{2l} = L_{2l-1}(x) - b'_{2l}x^{2l} \text{ say,}$$

hence

$$|\, B_{2l-1}(z') - B_{2l}(z')\,| \leqq \int_{-\infty}^{\infty} \int_{z'+L_{2l-1}(x)-b'_{2l}x^{2l}}^{z'+L_{2l-1}(x)} |w(x, y) + \gamma(x, y)\,|\, dy\, dx$$

$$\leqq Q_k\, b'_{2l} \leqq Q_k\,|\, z\,|\, n^{-l} < Q_k\,|\, z\,|\, n^{-\alpha_0}.$$

Therefore

$$|\, Pr\{Y - L_{2l-1}(X) \leqq z'\} - Pr\{Y - L_{2l}(X) \leqq z'\}\,| \leqq Q_k n^{-\alpha_0}$$

and so we obtain

$$F(z) = B(z') + \Lambda_k(1 + |\, z\,|^{2k-3})n^{-\alpha_0} \tag{40}$$

which is equivalent to (2) in the theorem stated in section 1.

Thus the theorem will be proved if the assertions regarding the form of $f(z)$ in (1) are shown to be true.

For this purpose we denote, as before, the terms of the order $n^{-\nu/2}$ in $\psi(it_1, it_2)$ and $\gamma(x, y)$ by ψ_ν, γ_ν respectively. Since the term in ψ_ν which yields a w_{pq} with the greatest q is U_3^ν, we have for every w_{pq} in γ_ν the condition $q \leqq 3\nu$. We expand $\int_{-\infty}^{\infty} \int_{-\infty}^{g(\lambda)} \gamma_\nu(x, y)dydx$ to $k - 3 - \nu$ terms, in which $f_{pq}(0)$, $f'_{pq}(0)$, $\cdots$, $f_{pq}^{(k-3-\nu)}(0)$ occur. In the integrand of $f_{pq}^{(k-3-\nu)}(0)$, e.g., the coefficients of each $w_{pq}(x, z)$ are polynomials in z and x of a total degree in z and x not exceeding that of $(g'(0))^{k-3-\nu}$, i.e., $2(k - 3 - \nu)$. Hence the expansion of γ_ν will give rise to terms of the form

$$z^s I_{pq}^t, \qquad q \leqq 3\nu, \qquad s + t = 2(k - 3 - \nu).$$

Such a term will yield a term $z^s \Phi^{(q+t)}$, which in turn yields the terms $\Phi^{(r)}$ with

$$r \leqq s + q + t \leqq 3\nu + 2(k - 3 - \nu) \leqq 3(k - 3),$$

Equality holds only when $\nu = k - 3$ and $q = 3(k - 3)$. But when $\nu = k - 3$, the term in question is

$$f_{0,3(k-3)}(0) = I^0_{0,3k-10} = \Phi^{(3k-10)}$$

Next, we see that ψ_ν contains $U_3, \cdots, U_{\nu+2}$. Since $f^{(k-3-\nu)}(0)$ is a polynomial of the $(k - 3 - \nu)$th degree in x, the expansion of γ_ν will yield $I^0_{pq}, \cdots, I^{k-3-\nu}_{pq}$. But $I^{k-3-\nu}_{pq} = 0$ if $p > k - 3 - \nu$, hence $p \leqq k - 3 - \nu$. Thus in ψ_ν we need only take account of the terms $(it_1)^p(it_2)^q$ with $p \leqq k - 3 - \nu$. Now if $j < k - 3 - \nu$, in U_j only $\alpha_3, \cdots, \alpha_{2(k-3-\nu)}$ occur. If $j \geqq k - 3 - \nu$, in the coefficient of a term $(it_1)^p(it_2)^q$ with $p \leqq k - 3 - \nu$ the greatest index of α is

$$2(k - 3 - \nu) + j - (k - 3 - \nu) = j + k - 3 - \nu \leqq k - 1$$

since $j \leqq \nu + 2$. Hence in the expansion of every γ only $\alpha_3, \cdots, \alpha_{k-1}$ occur. The proof of the theorem is completed.

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SOME IMPROVEMENTS IN SETTING LIMITS FOR THE EXPECTED NUMBER OF OBSERVATIONS REQUIRED BY A SEQUENTIAL PROBABILITY RATIO TEST

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Summary. Upper and lower limits for the expected number n of observations required by a sequential probability ratio test have been derived in a previous publication [1]. The limits given there, however, are far apart and of little practical value when the expected value of a single term z in the cumulative sum computed at each stage of the sequential test is near zero. In this paper upper and lower limits for the expected value of n are derived which will, in general, be close to each other when the expected value of z is in the neighborhood of zero. These limits are expressed in terms of limits for the expected values of certain functions of the cumulative sum Z_n at the termination of the sequential test.

In section 7 a general method is given for determining limits for the expected value of any function of Z_n.

1. Introduction. Let x be a random variable and let $f(x, \theta)$ be the elementary probability law of x involving an unknown parameter θ. Let H_0 denote the hypothesis that $\theta = \theta_0$, and H_1 the hypothesis that $\theta = \theta_1$, where θ_0 and θ_1 are given specified values. The sequential probability ratio test for testing H_0 against H_1, as defined in [1], is given as follows: Put

$$z_i = \log \frac{f(x_i, \theta_1)}{f(x_i, \theta_0)} \tag{1.1}$$

where x_i denotes the i-th observation on x. Two constants, a and b are chosen where $a > 0$ and $b < 0$. At each stage of the experiment, at the m-th trial for each positive integral value m, the cumulative sum

$$Z_m = z_1 + \cdots + z_m \tag{1.2}$$

is computed. Experimentation is continued as long as $b < Z_m < a$. The first time that Z_m does not lie between b and a, experimentation is terminated. The hypothesis H_1 is accepted if $Z_m \geqq a$, and H_0 is accepted if $Z_m \leqq b$.

Let n denote the smallest value of m for which Z_m does not lie between b and a. Then n is the number of observations required by the sequential test. The expected value of n is a function of the true parameter value θ and is denoted by $E_\theta(n)$.

Upper and lower limits for $E_\theta(n)$ have been derived in section 4 of [1]. These limits, however, are of little practical value when the expected value of

$$z = \log \frac{f(x, \theta_1)}{f(x, \theta_0)} \tag{1.3}$$

is in the neighborhood of zero, for they converge to $+\infty$ and $-\infty$, respectively, as the expected value of z approaches zero. It can be shown that the expected value of z is negative when $\theta = \theta_0$, and positive when $\theta = \theta_1$.[1] Thus, if the expected value of z is a continuous function of θ, there will be a value θ' between θ_0 and θ_1 such that the expected value of z is zero when $\theta = \theta'$. Hence, the limits for $E_\theta(n)$, as given in [1], are of no practical value when θ is near θ'.

The purpose of this paper is to derive upper and lower limits for $E_\theta(n)$ which will be, in general, close to each other when θ is in the neighborhood of θ'. Thus, it will generally be possible to obtain close limits for $E_\theta(n)$ over the whole range of θ, if the limits given here are used for values in a certain small interval containing θ', and the limits given in [1] are used when θ is outside this interval.

2. Notation. We shall use the following notations throughout the paper. For any random variable u, the symbol $E_\theta(u)$ will denote the expected value of u when θ is the true value of the parameter. The conditional expected value of u, under the restriction that some relationship R is fulfilled will be denoted by $E_\theta(u \mid R)$. The symbol $P(R \mid \theta)$ will denote the probability that the relationship R holds when θ is true.

The cumulative distribution function of z will be denoted by $F(z, \theta)$ when θ is the true value of the parameter. The moment generating function of z, when θ is true, will be denoted by $\varphi(t, \theta)$, i.e.

$$\varphi(t, \theta) = \int_{-\infty}^{\infty} e^{tz}\, dF(z, \theta). \tag{2.1}$$

3. Assumptions concerning the family of distribution functions $F(z, \theta)$. In this section we shall formulate two assumptions concerning $F(z, \theta)$ which will then be used to prove various lemmas and theorems. Since we are interested in values of θ near θ', we shall restrict the domain of θ to a finite closed interval I containing θ' in its interior. It will be understood throughout the paper that any statements concerning θ refer to the domain I, even if this is not explicitly stated.

ASSUMPTION 1. *The moment generating function $\varphi(t, \theta)$ exists for any point t in the complex plane and any value θ, and is a continuous function of θ.*

ASSUMPTION 2. *There exists a positive δ such that $P(e^z > 1 + \delta \mid \theta)$ and $P(e^z < 1 - \delta \mid \theta)$ have positive lower bounds with respect to θ.*

4. *Proof that $\varphi(t, \theta)$ is continuous in t and θ jointly and that all moments of z are continuous functions of θ.*[2] In this section we shall prove the following theorem:

[1] This follows easily from Lemma 1 in [1], p. 156.

[2] The original proof of the author was somewhat lengthy. The present proof was suggested by T. E. Harris.

THEOREM 4.1. *It follows from Assumption 1 that $\varphi(t, \theta)$ is continuous in t and θ jointly and all moments of z are continuous functions of θ.*

PROOF: First we show that $\varphi(t, \theta)$ is a bounded function of t and θ in the domain $|t| \leqq t_0$, for any finite positive value t_0. Clearly,

$$0 \leqq |\varphi(t, \theta)| \leqq 2[\varphi(t_0, \theta) + \varphi(-t_0, \theta)] \tag{4.1}$$

for all values t for which $|t| \leqq t_0$. The boundedness of $\varphi(t_0, \theta)$ and $\varphi(-t_0, \theta)$ follows from Assumption 1. Hence $\varphi(t, \theta)$ is a bounded function of θ and t over any bounded t-domain.

Let $\{t_m, \theta_m\}$ $(m = 1, 2, \cdots, \text{ad inf.})$ be a sequence of pairs converging to the pair (t', θ'). We have

$$\varphi(t_m, \theta_m) - \varphi(t', \theta') = [\varphi(t_m, \theta_m) - \varphi(t', \theta_m)] + [\varphi(t', \theta_m) - \varphi(t', \theta')]. \tag{4.2}$$

The second expression in brackets converges to zero by continuity in θ. Thus the first part of Theorem 4.1 is proved if we show that

$$\lim_{m \to \infty} [\varphi(t_m, \theta_m) - \varphi(t', \theta_m)] = 0. \tag{4.3}$$

It follows from Assumption 1 that for any given θ, $\varphi(t, \theta)$ is an analytic function with no singularities in any finite t-domain. Hence we can expand $\varphi(t_m, \theta_m)$ in a Taylor series around $t = t'$, i.e.

$$\varphi(t_m, \theta_m) - \varphi(t', \theta_m) = \sum_{k=1}^{\infty} \frac{1}{k!} \left(\frac{\partial^k \varphi(t, \theta_m)}{\partial t^k} \Big|_{t=t'} \right) (t_m - t')^k. \tag{4.4}$$

Let r be a given positive value. Because of the boundedness of $\varphi(t, \theta)$ in any finite t-domain, there exists a constant M such that $|\varphi(t, \theta)| < M$ for all θ and for all t in the domain $|t - t'| \leqq r$. From the Cauchy integral formula for an analytic function it follows that

$$\frac{1}{k!} \left| \frac{\partial^k \varphi(t, \theta_m)}{\partial t^k} \Big|_{t=t'} \right| \leqq \frac{M}{r^k}. \tag{4.5}$$

From (4.4) and (4.5) we obtain

$$|\varphi(t_m, \theta_m) - \varphi(t', \theta_m)| \leqq M \sum_{k=1}^{\infty} \frac{|t_m - t'|^k}{r^k}. \tag{4.6}$$

Equation (4.3) is an immediate consequence of (4.6). This proves the first half of Theorem 4.1.

Let C be a circle in the complex t-plane with finite radius and center at the origin. According to the Cauchy integral formula we have

$$\frac{1}{2\pi i} \int_C \frac{\varphi(t, \theta)}{t^{k+1}} \, dt = \frac{1}{k!} \frac{\partial^k \varphi(t, \theta)}{\partial t^k} \Big|_{t=0} = \frac{1}{k!} E_\theta(z^k). \tag{4.7}$$

Since $\varphi(t, \theta)$ is continuous in t and θ jointly, the integral on the left hand side of (4.7) is a continuous function of θ. This proves the second half of Theorem 4.1.

5. Some lemmas. In this section we shall prove several lemmas which will then be used to derive the results contained in sections 6 and 8.

LEMMA 5.1. *It follows from assumptions* 1 *and* 2 *that for any given* θ *the equation in* t

$$\varphi(t, \theta) = 1 \tag{5.1}$$

has exactly two real roots, one of which is zero. The other real root is different from zero if $E_\theta(z) \neq 0$. *If* $E_\theta(z) = 0$, *both roots are equal to zero, i.e., zero is a double root of* (5.1).

This lemma is essentially the same as Lemma 2 in [2] and the proof is therefore omitted.[3]

Let $h(\theta)$ denote the non-zero root of (5.1), if $E_\theta(z) \neq 0$. If $E_\theta(z) = 0$, we put $h(\theta) = 0$.

In what follows the variable t will be restricted to **real** values, unless the contrary is explicitly stated.

LEMMA 5.2. *It follows from assumptions* 1 *and* 2 *that* $h(\theta)$ *is a continuous function of* θ.

PROOF: It follows from assumption 2 that

$$\lim_{t \to \pm\infty} \varphi(t, \theta) = +\infty \tag{5.2}$$

uniformly in θ. Hence, since by definition

$$\varphi[h(\theta), \theta] = 1$$

identically in θ, $h(\theta)$ must be a bounded function of θ.

Let $\{\theta_m\}$ be a sequence of parameter values which converges to θ^*. From Theorem 4.1 it follows that

$$\lim_{m \to \infty} [\varphi(t, \theta_m) - \varphi(t, \theta^*)] = 0 \tag{5.3}$$

uniformly in t over any finite interval. Since $h(\theta)$ is bounded, we obtain from (5.3)

$$\lim_{m \to \infty} \{\varphi[h(\theta_m), \theta_m] - \varphi[h(\theta_m), \theta^*]\} = 0. \tag{5.4}$$

Since $\varphi[h(\theta_m), \theta_m] = 1$, it follows from (5.4) that

$$\lim_{m \to \infty} \varphi[h(\theta_m), \theta^*] = 1.$$

It follows from assumption 1 that for any limit point h of the bounded sequence $\{h(\theta_m)\}$ ($m = 1, 2, \cdots$, ad inf.) we have

[3] Condition IV of Lemma 2 in [2] is not postulated here, since the validity of this condition is implied by assumption 1. Condition IV could have been omitted also in [2], since it follows from condition III.

$$\varphi(h, \theta^*) = 1 \tag{5.5}$$

If $h(\theta^*) = 0$, then equation $\varphi(t, \theta^*) = 1$ has the only root $t = 0$. Consequently, all limit points of $\{h(\theta_m)\}$ must be equal to zero, that is

$$\lim_{m \to \infty} h(\theta_m) = 0 \quad \text{if} \quad h(\theta^*) = 0. \tag{5.6}$$

Now let us assume that $h(\theta^*) \neq 0$. Since the second derivative of $\varphi(t, \theta)$ with respect to t is positive, it can be seen that $\varphi(t, \theta) < 1$ for values t in the open interval $(0, h(\theta))$, and $\varphi(t, \theta) > 1$ for any t outside the closed interval $[0, h(\theta)]$. Hence, $\varphi(t, \theta) < 1$ implies that $|h(\theta)| > |t|$ and $h(\theta)$ and t have the same sign. Now let t_0, be a value in the open interval $(0, h(\theta^*))$. Then we have

$$\varphi(t_0, \theta^*) < 1 \tag{5.7}$$

It follows from assumption 1 that

$$\varphi(t_0, \theta_m) < 1 \tag{5.8}$$

for sufficiently large m. Hence $h(\theta_m)$ and t_0 have the same sign and

$$|h(\theta_m)| > |t_0| \tag{5.9}$$

Inequality (5.9) implies that zero cannot be a limit point of the sequence $\{h(\theta_m)\}$. Since $\varphi(t, \theta^*) = 1$ has only the roots $t = 0$ and $t = h(\theta^*)$, it follows from (5.5) that the sequence $\{h(\theta_m)\}$ cannot have a limit point different from $h(\theta^*)$. Thus,

$$\lim_{m \to \infty} h(\theta_m) = h(\theta^*) \tag{5.10}$$

and Lemma 5.2 is proved.

LEMMA 5.3. *It follows from assumption* 1 *that for any given* t, $E_\theta(e^{|zt|})$ *is a bounded function of* θ.

PROOF: We have

$$E_\theta(e^{|tz|}) \leqq E_\theta(e^{tz} + e^{-tz}) = \varphi(t, \theta) + \varphi(-t, \theta) \tag{5.11}$$

It follows from assumption 1 that $\varphi(t, \theta)$ and $\varphi(-t, \theta)$ are bounded functions of θ. Hence Lemma 5.3 is proved.

LEMMA 5.4. *Let* θ' *be a value of* θ *such that* $E_{\theta'}(z) = 0$, *but* $E_\theta(z) \neq 0$ *for all* $\theta \neq \theta'$ *in an open interval containing* θ'. *It follows from assumptions* 1 *and* 2 *that*

$$\lim_{\theta \to \theta'} \left(-\frac{2E_\theta(z)}{h(\theta)} \right) = E_{\theta'}(z^2). \tag{5.12}$$

PROOF: We have

$$e^{h(\theta)z} = 1 + h(\theta)z + \frac{[h(\theta)]^2}{2} z^2 + \frac{[h(\theta)]^3}{6} z^3 e^{uh(\theta)z} \tag{5.13}$$

where $0 \leqq u \leqq 1$. Hence

$$(5.14)\quad E_\theta(e^{h(\theta)z}) = 1 + h(\theta)E_\theta(z) + \frac{[h(\theta)]^2}{2} E_\theta(z^2) + \frac{[h(\theta)]^3}{6} E_\theta(z^3 e^{uh(\theta)z}).$$

Since $E_\theta(e^{h(\theta)z}) = 1$, we obtain from (5.14)

$$(5.15)\qquad h(\theta)E_\theta(z) + \frac{[h(\theta)]^2}{2} E_\theta(z^2) + \frac{[h(\theta)]^3}{6} E_\theta(z^3 e^{uh(\theta)z}) = 0.$$

We shall consider only values θ for which $h(\theta) \neq 0$. For such values of θ, also $E_\theta(z) \neq 0$. Dividing (5.15) by $h(\theta)E_\theta(z)$, we obtain

$$(5.16)\qquad 1 + \frac{h(\theta)}{2E_\theta(z)}\left[E_\theta(z^2) + \frac{h(\theta)}{3} E_\theta(z^3 e^{uh(\theta)z})\right] = 0.$$

Let t_0 be an upper bound of $|h(\theta)|$ with respect to θ. Then for a suitably chosen constant C we have

$$(5.17)\qquad |z^3 e^{uh(\theta)z}| < Ce^{|t_0 z|}.$$

From this and Lemma 5.3 it follows that $E_\theta(z^3 e^{uh(\theta)z})$ is a bounded function of θ.

Because of the continuity of $h(\theta)$ we have

$$(5.18)\qquad \lim_{\theta \to \theta'} h(\theta) = 0.$$

Lemma 5.4 follows from (5.16), (5.18), the boundedness of $E_\theta(z^3 e^{uh(\theta)z})$ and the fact that $E_\theta(z^2)$ is a continuous function of θ and $E_{\theta'}(z^2) > 0$.

LEMMA 5.5. *From assumptions* 1 *and* 2 *it follows that for any given* t, $E_\theta(e^{|tz_n|})$ *exists and is a bounded function of* θ.

PROOF: It is sufficient to show that $E_\theta(e^{tz_n})$ is a bounded function of θ for any t, since

$$(5.19)\qquad e^{|tz_n|} \leqq e^{tz_n} + e^{-tz_n}$$

Clearly, e^{tz_n} lies between $e^{bt+z_n t}$ and $e^{at+z_n t}$. Hence Lemma 5.5 is proved if we show that $E_\theta(e^{z_n t})$ is a bounded function of θ.

It follows from Assumption 2 that there exists a positive integer k and a positive constant g such that

$$(5.20)\qquad P(|z_1 + \cdots + z_k| \geqq a - b \,|\, \theta) \geqq g$$

for all θ. For any positive integer m and for any real values $\lambda_1 < \lambda_2$ we have

$$(5.21)\qquad \frac{P[(m-1)k < n \leqq mk \,|\, \theta]}{P[(m-1)k < n \,|\, \theta]} \geqq g \qquad (m = 1, 2, \cdots, \text{ad inf.})$$

and

$$(5.22)\qquad \frac{P[(m-1)k < n \leqq mk \;\&\; \lambda_1 \leqq z_n < \lambda_2 \,|\, \theta]}{P[(m-1)k < n \,|\, \theta]} \leqq 1 - [1 - P(\lambda_1 \leqq z < \lambda_2 \,|\, \theta)]^k.$$

Hence

$$\frac{P[(m-1)k < n \leqq mk \,\&\, \lambda_1 \leqq z_n < \lambda_2 \,|\, \theta]}{P[(m-1)k < n \leqq mk \,|\, \theta]} \leqq \frac{1 - [1 - P(\lambda_1 \leqq z < \lambda_2 \,|\, \theta]^k}{g}. \tag{5.23}$$

Multiplying (5.23) by $P[(m-1)k < n \leqq mk \,|\, \theta]$ and summing with respect to m we obtain

$$P(\lambda_1 \leqq z_n < \lambda_2 \,|\, \theta) \leqq \frac{1 - [1 - P(\lambda_1 \leqq z < \lambda_2 \,|\, \theta)]^k}{g}. \tag{5.24}$$

From (5.24) it follows readily that

$$\frac{P(\lambda_1 \leqq z_n < \lambda_2 \,|\, \theta)}{P(\lambda_1 \leqq z < \lambda_2 \,|\, \theta)} \tag{5.25}$$

is a bounded function of λ_1, λ_2 and θ. Let A be an upper bound of the ratio (5.25). Then

$$E_\theta(e^{tz_n}) \leqq AE_\theta(e^{tz}) = A\varphi(t, \theta). \tag{5.26}$$

Because of Assumption 1, $\varphi(t, \theta)$ is a bounded function of θ. Hence also $E_\theta(e^{tz_n})$ is bounded and Lemma 5.5 is proved.

6. The limiting value of $E_\theta(n)$ when θ approaches a value θ' for which $E_{\theta'}(z) = 0$. In this section we shall prove the following theorem:

THEOREM 6.1. *Let θ' be a value of θ such that $E_{\theta'}(z) = 0$, but $E_\theta(z) \neq 0$ for all $\theta \neq \theta'$ in an open interval containing θ'. If assumptions* 1 *and* 2 *hold, we have*

$$\lim_{\theta \to \theta'} \left[E_\theta(n) - \frac{E_\theta(Zn^2)}{E_{\theta'}(z^2)} \right] = 0. \tag{6.1}$$

PROOF: Consider the Taylor expansion

$$e^{h(\theta)z_n} = 1 + h(\theta)Z_n + \frac{[h(\theta)]^2}{2} Z_n^2 + \frac{[h(\theta)]^3}{6} Z_n^3 e^{\lambda h(\theta) z_n} \tag{6.2}$$

where $0 \leqq \lambda \leqq 1$. It was shown in [2] (p. 286) that

$$E_\theta e^{h(\theta) z_n} = 1. \tag{6.3}$$

Hence, taking expected values on both sides of (6.2), we obtain

$$h(\theta)E_\theta(Z_n) + \frac{[h(\theta)]^2}{2} E_\theta(Z_n^2) + \frac{[h(\theta)]^3}{6} E_\theta(Z_n^3 e^{\lambda h(\theta) z_n}) = 0. \tag{6.4}$$

We consider only values of θ for which $E_\theta(z) \neq 0$. For such values, also $h(\theta) \neq 0$. Thus, we can divide both sides of (6.4) by $h(\theta)E_\theta(z)$. We then obtain

$$\frac{E_\theta(Z_n)}{E_\theta(z)} + \frac{h(\theta)}{2E_\theta(z)}\left[E_\theta Z_n^2 + \frac{h(\theta)}{3} E_\theta(Z_n^3 e^{\lambda h(\theta) Z_n})\right] = 0. \tag{6.5}$$

It was shown in [1] (p. 142) that

$$E_\theta(n) = \frac{E_\theta(Z_n)}{E_\theta(z)}. \tag{6.6}$$

Hence

$$E_\theta(n) + \frac{h(\theta)}{2E_\theta(z)}\left[E_\theta(Z_n^2) + \frac{h(\theta)}{3} E_\theta(Z_n^3 e^{\lambda h(\theta) Z_n})\right] = 0. \tag{6.7}$$

Let t_0 be an upper bound of $|h(\theta)|$. Then for a properly chosen constant C we have

$$|Z_n^3 e^{\lambda h(\theta) Z_n}| \leqq C e^{|t_0 Z_n|} \tag{6.8}$$

From this and Lemma 5.5 it follows that $E_\theta(Z_n^3 e^{\lambda h(\theta) Z_n})$ is a bounded function of θ. Since $\lim_{\theta \to \theta'} h(\theta) = 0$ and $E_\theta(Z_n^2)$ has a positive lower bound, Theorem 6.1 follows from 6.7, Lemma 5.4 and Theorem 4.1.

If $\lim_{\theta \to \theta'} E_\theta Z_n^2 = E_{\theta'} Z_n^2$, Theorem 6.1 gives[4]

$$E_{\theta'}(n) = \frac{E_{\theta'}(Z_n^2)}{E_{\theta'}(z^2)}. \tag{6.9}$$

Limits for $E_{\theta'}(n)$ can be obtained by computing limits for $E_\theta(Z_n^2)$. In the next section we shall give a general method for obtaining limits for $E_\theta[\psi(Z_n)]$, where $\psi(Z_n)$ is any function of Z_n.

7. Determination of lower and upper limits for the expected value of any function of Z_n. Let $\psi(Z_n)$ be a function of Z_n. Limits for $E_\theta[\psi(Z_n)]$ may be determined as follows: First we determine limits for $E_\theta[\psi(Z_n) \mid Z_n \geqq a]$. Let r be a positive variable. Clearly, for any given value r we have

$$E_\theta[\psi(Z_n) \mid Z_{n-1} = a - r \text{ and } Z_n \geqq a] = E_\theta[\psi(a - r + z) \mid z \geqq r] \tag{7.1}$$

From (7.1) we obtain the limits

$$\begin{aligned} \text{g.l.b.}_{0<r<a-b} E_\theta[\psi(a - r + z) \mid z \geqq r] &\leqq E_\theta[\psi(Z_n) \mid Z_n \geqq a] \\ &\leqq \text{l.u.b.}_{0<r<a-b} E_\theta[\psi(a - r + z) \mid z \geqq r]. \end{aligned} \tag{7.2}$$

Limits for $E_\theta[\psi(Z_n) \mid Z_n \leqq b]$ can be obtained in a similar way. Again, let r be a positive variable. For any value of r we have

$$E_\theta[\psi(Z_n) \mid Z_n \leqq b \text{ and } Z_{n-1} = b + r] = E_\theta[\psi(b + r + z) \mid z \leqq -r \tag{7.3}$$

Hence we obtain the limits

[4] The validity of (6.9) was shown by the author [3] using an entirely different method.

$$\text{g.l.b.}_{0<r<a-b} E_\theta[\psi(b+r+z) \mid z \leqq -r] \leqq E_\theta[\psi(Z_n) \mid Z_n \leqq b] \leqq \text{l.u.b.}_{0<r<a-b} E_\theta[\psi(b+r+z) \mid z \leqq -r]. \tag{7.4}$$

Since

$$E_\theta[\psi(Z_n)] = P(Z_n \geqq a)E_\theta[\psi(Z_n) \mid Z_n \geqq a] + P(Z_n \leqq b)E_\theta[\psi(Z_n) \mid Z_n \leqq b], \tag{7.5}$$

a lower (upper) limit for $E_\theta[\psi(Z_n)]$ can be obtained, by replacing the conditional expected values on the right hand side of (7.5) by their lower (upper) limits given in (7.2) and (7.4).

8. Limits for $E_\theta(n)$ when $h(\theta)$ is near but unequal to zero. Let θ' be a value of θ for which $h(\theta') = 0$. In this section we shall derive limits for $E_\theta(n)$ which will generally be close to each other for values θ in a small neighborhood of θ'.

From equation (6.7) we obtain

$$E_\theta(n) = -\frac{h(\theta)}{2E_\theta(z)}\left[E_\theta Z_n^2 + \frac{h(\theta)}{3} E_\theta(Z_n^3 e^{\lambda h(\theta) Z_n})\right] \tag{8.1}$$

where $0 \leqq \lambda \leqq 1$. Thus, limits for $E_\theta(n)$ can be obtained by deriving limits for $E_\theta Z_n^2$ and $E_\theta(Z_n^3 e^{\lambda h(\theta) Z_n})$. Limits for $E_\theta Z_n^2$ can be obtained by using the method described in section 7.

If θ is near θ', any crude limits for $E_\theta(Z_n^3 e^{\lambda h(\theta) Z_n})$ will serve the purpose, since, as has been shown in section 6, $E_\theta(Z_n^3 e^{\lambda h(\theta) Z_n})$ is bounded and $\lim_{\theta \to \theta'} h(\theta) = 0$.

Limits for $E_\theta(Z_n^3 e^{\lambda h(\theta) Z_n})$ can be obtained as follows: For simplicity, let us assume that $h(\theta) > 0$. Then

$$Z_n^3 \leqq Z_n^3 e^{\lambda h(\theta) Z_n} \leqq Z_n^3 e^{h(\theta) Z_n} \qquad (h(\theta) > 0) \tag{8.2}$$

Thus, to determine limits for $E_\theta(Z_n^3 e^{\lambda h(\theta) Z_n})$, it is sufficient to determine a lower limit for $E_\theta(Z_n^3)$ and an upper limit for $E_\theta(Z_n^3 e^{h(\theta) Z_n})$. The latter limits may be derived by using the method given in section 7.

If $h(\theta) < 0$, we have

$$Z_n^3 \geqq Z_n^3 e^{\lambda h(\theta) Z_n} \geqq Z_n^3 e^{h(\theta) Z_n} \tag{8.3}$$

and a similar procedure will yield the desired limits for $E_\theta(Z_n^3 e^{\lambda h(\theta) Z_n})$.

It should be emphasized that the limits of $E_\theta(n)$, as given in this section, can be expected to be close only if $h(\theta)$ is near zero. For values of θ for which $h(\theta)$ is not near zero, the limits of $E_\theta(n)$ given in [1] can be used.

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THE EFFICIENCY OF THE MEAN MOVING RANGE

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Summary. In studying the variation of a variable subject to erratic trend effects, it is customary to employ as a measure of variation a statistic that eliminates most of such effects. It is shown in this paper that the statistic $w = \sum_1^{n-1} |x_{i+1} - x_i| \sqrt{\pi}/2(n-1)$ is nearly as efficient as the statistic $\delta^2 = \sum_1^{n-1} (x_{i+1} - x_i)^2/(n-1)$ that is customarily employed. The asymptotic variance of w is obtained by integration techniques; the proof of the asymptotic normality of w is based upon a theorem of S. Bernstein on the asymptotic distribution of sums of dependent variables. The method of proof is sufficiently general to prove the asymptotic normality of w, and of δ^2, for x having a distribution for which the third absolute moment exists.

1. Introduction. Let $x_1, x_2, \cdots, x_n$ denote a random sample of size n from a population with a continuous distribution function $f(x)$. If a measure of the variability of x is desired, it is customary to select the familiar statistic

$$s^2 = \frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n-1}, \tag{1}$$

or its positive square root s, as an estimate of the corresponding theoretical measure of variability.

If, however, it is known that the variable x is subject to trend effects and that $f(x)$ represents the distribution of x without such effects, then s^2 will not serve as a satisfactory measure of variability about the trend. In order to eliminate the influence of trends, it is helpful to employ statistics that capitalize on the time order relationships of the observations. There are several statistics of this type available, although most of them make no pretense of completely eliminating trend effects, even if the trend is linear.

Perhaps the best known among statistics of the desired type is the mean square successive difference,

$$\delta^2 = \frac{\sum_{i=1}^{n-1} (x_{i+1} - x_i)^2}{n-1}. \tag{2}$$

This measure of variation has been studied extensively in recent years. Among the results of these investigations is a determination [1] of the efficiency of $\delta^2/2$ as an estimate of σ^2 for a normally distributed variable when no trend exists.

A closely related measure of variation that is not so well known is the mean moving range of successive pairs of observations,

$$w = \frac{\sum_{i=1}^{n-1} | x_{i+1} - x_i |}{n - 1}. \tag{3}$$

Although w appears [1] to have been used by ballisticians, very little seems to be known concerning the relative merits of δ^2 and w. Since w is considerably easier to calculate than δ^2, it would be preferred to δ^2 for applications in which computational advantages are important. However, one would hardly allow such advantages to dominate a choice unless δ^2 and w were about equally efficient as estimates of variation.

The purpose of this paper is to determine the efficiency of w and to study efficiency properties of generalizations of w.

2. Definition of efficiency. The definition that will be used in this paper [2] may be stated in the following manner. Let θ be a parameter, or a function of parameters, of the distribution function $f(x)$. Let T be a statistic for which there exists a number μ such that

$$t = \sqrt{n}\,(T - \theta)$$

is asymptotically normally distributed with zero mean and variance μ^2. Let T' be any other statistic for which there exists a number μ' such that

$$t' = \sqrt{n}\,(T' - \theta)$$

is asymptotically normally distributed with zero mean and variance μ'^2. Then T is said to be an efficient estimate of θ provided that $\mu \leq \mu'$ for all possible choices of T', and the efficiency of any particular T' is defined to be

$$\mathcal{E}_{T'} = \left(\frac{\mu}{\mu'}\right)^2. \tag{4}$$

In order to determine the efficiency of a statistic, it is therefore necessary to first demonstrate its asymptotic normal distribution and then calculate its asymptotic variance. This order of procedure will be reversed in the following determination of the efficiency of w.

3. Variance of w. Let x be normally distributed with zero mean and unit variance. Then the mean of w, where w is given by (3), may be evaluated as follows:

$$\begin{aligned} E(w) &= E\,|\,x_2 - x_1\,| \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int |\,x_2 - x_1\,|\, e^{-\frac{1}{2}(x_1^2+x_2^2)}\,dx_1\,dx_2 \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-(x_2^2/2)}\left[\int_{-\infty}^{x_2}(x_2 - x_1)e^{-(x_1^2/2)}\,dx_1 + \int_{x_2}^{\infty}(x_1 - x_2)e^{-(x_1^2/2)}\,dx_1\right]dx_2 \end{aligned}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-(x_2^2/2)}\left[x_2\int_{-\infty}^{x_2} e^{-(x_1^2/2)}\,dx_1 - x_2\int_{x_2}^{\infty} e^{-(x_1^2/2)}\,dx_1 + 2e^{-(x_2^2/2)}\right]dx_2$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-(x_2^2/2)}\cdot 2\left[x_2\int_{0}^{x_2} e^{-(x_1^2/2)}\,dx_1 + e^{-(x_2^2/2)}\right]dx_2 .$$

If integration by parts is performed on the first integral with

$$u = \int_0^{x_2} e^{-(x_1^2/2)}\,dx_1 \qquad \text{and} \qquad dv = x_2\, e^{-(x_2^2/2)}\,dx_2 ,$$

the uv term will vanish at both limits and $E(w)$ will reduce to

$$E(\mathrm{w}) = \frac{2}{\pi}\int_{-\infty}^{\infty} e^{-x_2^2}\,dx_2 = \frac{2}{\sqrt{\pi}}. \tag{5}$$

This result could have been obtained more easily by other methods, but some of the integrals involved will be needed later.

For the purpose of computing the second moment of w, it is convenient to separate the independent and dependent product terms of w^2. Since there are $2(n-2)$ of the latter, $E(\mathrm{w}^2)$ may be expressed in the form

$$(n-1)^2E(\mathrm{w}^2) = (n-1)E\,|\,x_2 - x_1\,|^2 + 2(n-2)E\,|\,x_2 - x_1\,||\,x_3 - x_2\,| + (n-2)(n-3)E^2\,|\,x_2 - x_1\,|.$$

But

$$E\,|\,x_2 - x_1\,|^2 = E(x_2 - x_1)^2 = E(x_2^2) + E(x_1^2) = 2.$$

Consequently, because of (5),

$$(n-1)^2E(\mathrm{w}^2) = 2(n-2)E\,|\,x_2 - x_1\,||\,x_3 - x_2\,| + 2(n-1) + 4(n-2)(n-3)/\pi. \tag{6}$$

Now consider the evaluation of the product term

$$E\,|\,x_2 - x_1\,||\,x_3 - x_2\,| = (2\pi)^{-\frac{3}{2}}\int\int_{-\infty}^{\infty}\int |\,x_2 - x_1\,||\,x_3 - x_2\,|e^{-\frac{1}{2}(x_1^2+x_2^2+x_3^2)}\,dx_1\,dx_2\,dx_3 .$$

By means of the expressions that were used to give (5), this triple integral may be reduced in the following manner:

$$E\,|\,x_2 - x_1\,||\,x_3 - x_2\,| = (2\pi)^{-\frac{3}{2}}\int_{-\infty}^{\infty}\int |\,x_3 - x_2\,|e^{-\frac{1}{2}(x_2^2+x_3^2)}\cdot 2\left[x_2\int_0^{x_2} e^{-(x_1^2/2)}\,dx_1 + e^{-(x_2^2/2)}\right]dx_3\,dx_2$$

$$= (2\pi)^{-\frac{1}{2}}\int_{-\infty}^{\infty} e^{-(x_2^2/2)}\cdot 4\left[x_2\int_0^{x_2} e^{-(x_1^2/2)}\,dx_1 + e^{-(x_2^2/2)}\right]^2 dx_2$$

$$= 4(2\pi)^{-\frac{1}{2}}\int_{-\infty}^{\infty} e^{-(x_2^2/2)}\left[x_2^2\left(\int_0^{x_2} e^{-(x_1^2/2)}\,dx_1\right)^2 + 2x_2e^{-(x_2^2/2)}\int_0^{x_2} e^{-(x_1^2/2)}\,dx_1 + e^{-x_2^2}\right]dx_2 .$$

These three integrals, without their constant factors, will be denoted by I_1, I_2, and I_3, respectively. I_1 may be evaluated by integrating by parts with

$$u = x_2 \left(\int_0^{x_2} e^{-(x_1^2/2)}\, dx_1 \right)^2 \quad \text{and} \quad dv = x_2 e^{-(x_2^2/2)}\, dx_2 .$$

The uv term will vanish at both limits; consequently

$$\begin{aligned} I_1 &= \int_{-\infty}^{\infty} e^{-(x_2^2/2)} \left[2x_2 e^{-(x_2^2/2)} \int_0^{x_2} e^{-(x_1^2/2)}\, dx_1 + \left(\int_0^{x_2} e^{-(x_1^2/2)}\, dx_1 \right)^2 \right] dx_2 \\ &= 2 \int_{-\infty}^{\infty} x_2 e^{-x_2^2} \int_0^{x_2} e^{-(x_1^2/2)}\, dx_1\, dx_2 + \int_{-\infty}^{\infty} e^{-(x_2^2/2)} \left(\int_0^{x_2} e^{-(x_1^2/2)}\, dx_1 \right)^2 dx_2 . \end{aligned} \tag{7}$$

The first of these two integrals may be evaluated in the same manner as the first integral preceding (5). The second integral may be evaluated by making the change of variable

$$u = \int_0^{x_2} e^{-(x_1^2/2)}\, dx_1 .$$

As a result of such manipulations,

$$I_1 = \frac{\sqrt{6\pi}}{3} + \frac{\pi\sqrt{2\pi}}{6} .$$

It will be observed that I_2 is the same as the first integral of (7) and that I_3 is available in tables; hence

$$\begin{aligned} &E\,|\,x_2 - x_1\,|\,|\,x_3 - x_2\,| \\ &\quad = 4(2\pi)^{-1} \left[\frac{\sqrt{6\pi}}{3} + \frac{\pi\sqrt{2\pi}}{6} + \frac{\sqrt{6\pi}}{3} + \frac{\sqrt{6\pi}}{3} \right] = \tfrac{1}{3} + \frac{2\sqrt{3}}{\pi} . \end{aligned} \tag{8}$$

If (8) is substituted in (6), $E(\mathrm{w}^2)$ will reduce to

$$E(\mathrm{w}^2) = \frac{2(n-2)}{(n-1)^2} \left[\tfrac{1}{3} + \frac{2\sqrt{3}}{\pi} \right] + \frac{2}{n-1} + \frac{4(n-2)(n-3)}{\pi(n-1)^2} . \tag{9}$$

Since $\sigma_{\mathrm{w}}^2 = E(\mathrm{w}^2) - E^2(\mathrm{w})$, (9) and (5) will yield the following desired variance of w,

$$\sigma_{\mathrm{w}}^2 = \frac{2}{(n-1)^2} \left[\left(\tfrac{4}{3} + \frac{2\sqrt{3} - 6}{\pi} \right) n + \left(\frac{10 - 4\sqrt{3}}{\pi} - \tfrac{5}{3} \right) \right] . \tag{10}$$

4. Efficiency of w. Now let x be normally distributed with mean m and variance σ^2. Then the mean of w as given by (5) will be multiplied by σ and the variance of w as given by (10) will be multiplied by σ^2; consequently $z = \mathrm{w}\sqrt{\pi}/2$ will serve as an unbiased estimate of σ. In the next section it will be shown that

$$t' = \sqrt{n}(z - \sigma)$$

possesses an asymptotic normal distribution. From (10) and section 2, it therefore follows that the asymptotic variance, μ'^2, that is needed to determine the efficiency of z is given by

$$\mu'^2 = \frac{\pi}{4} 2\left(\frac{4}{3} + \frac{2\sqrt{3} - 6}{\pi}\right) = \frac{2\pi}{3} + \sqrt{3} - 3.$$

Now it is known that for x normally distributed s, as defined by (1), is an efficient estimate of σ with $\mu^2 = \frac{1}{2}$; consequently, because of (4), the efficiency of z as an estimate of σ is given by

$$\mathcal{E}_z = \frac{1}{2\left(\frac{2\pi}{3} + \sqrt{3} - 3\right)} \doteq .605. \tag{11}$$

In [1] it was shown that for x normally distributed $\delta^2/2$ was an unbiased estimate of σ^2 and, assuming the normality of its asymptotic distribution, that the efficiency of $\delta^2/2$ as an estimate of σ^2 was 2/3. Thus, $z = \mathrm{w}\sqrt{\pi}/2$ possesses very nearly the same efficiency as a measure of variation of a normal variable as $\delta^2/2$ does.

5. Asymptotic distribution of mean moving ranges. Although the efficiency obtained in the preceding section requires for its validity merely a demonstration that for x normally distributed w possesses an asymptotic normal distribution, it will be shown in this section that general mean moving ranges of a continuous variable x possess asymptotic normal distributions provided only that x possesses a third absolute moment.

Let r_i denote the range of the observations from x_i to x_{i+k-1}. Then the variable

$$W = \frac{r_1 + r_2 + \cdots + r_{r-k+1}}{n - k + 1} \tag{12}$$

will represent a generalized mean moving range, of which w will be a special case when $k = 2$.

A proof of the asymptotic property of W can be constructed as an application of a general theorem of S. Bernstein [3]. Since his theorem is long and involves much explanation of notation, a simplified version of it that is sufficient to cover this application, and indeed many similar applications, will be given.

Let $y_1, y_2, \cdots, y_m$ denote m variables for which the third absolute moments are bounded and let

$$S_m = y_1 + y_2 + \cdots + y_m.$$

Then Bernstein's theorem implies that if there exist constants c_1, c_2, c_3, and c_4 such that

(a) $$c_1 m < \sigma^2_{s_m} < c_2 m,$$

and

(b) y_i and y_{i+g} are independently distributed for

$$g > c_3 m^{c_4},\ c_4 < \tfrac{1}{2},$$

then

$$\frac{S_m - E(S_m)}{\sigma_{S_m}}$$

possesses an asymptotic normal distribution with zero mean and unit variance.

Consider the application of this theorem to $R = (n - k + 1)W$. The variance of R may be expressed in compact form by means of the techniques of section 3. Since r_i is the range of k consecutive observations, it is clear that

$$E(r_i r_{i+g}) = E^2(r_1)$$

if $g \geq k$. Furthermore, for subscripts for which it is defined, $E(r_i r_{i+g})$ will be independent of i. These two properties may be used to collect terms in the expansion of $E(R^2)$ to give

$$E(R^2) = (n - k + 1)E(r_1^2) + 2\sum_{i=0}^{k-2}(n - k - i)E(r_1 r_{2+i}) + (n - 2k + 1)(n - 2k + 2)E^2(r_1).$$

Consequently,

$$\sigma_R^2 = (n - k + 1)E(r_1^2) + 2\sum_{i=0}^{k-2}(n - k - i)E(r_1 r_{2+i}) + [n(1 - 2k) + (k - 1)(3k - 1)]E^2(r_1). \tag{13}$$

From the definition of the correlation coefficient and the fact that a correlation coefficient cannot exceed one, it follows that

$$\begin{aligned} E(r_1 r_{2+i}) &\leq E(r_1)E(r_{2+i}) + \sigma_{r_1}\sigma_{r_{2+i}} \\ &\leq E^2(r_1) + \sigma_{r_1}^2 . \end{aligned}$$

If this inequality is applied to (13),

$$\begin{aligned} \sigma_R^2 &\leq (n - k + 1)E(r_1^2) + (k - 1)(2n - 3k + 2)[E^2(r_1) + \sigma_{r_1}^2] + [n(1 - 2k) + (k - 1)(3k - 1)]E^2(r_1) \\ &\leq (n - k + 1)[E(r_1^2) - E^2(r_1)] + (k - 1)(2n - 3k + 2)\sigma_{r_1}^2 \\ &\leq [n(2k - 1) - (k - 1)(3k - 1)]\sigma_{r_1}^2 \\ &\leq 2k\sigma_{r_1}^2(n - k + 1). \end{aligned}$$

Thus, for a fixed k the right inequality in (a) of Bernstein's modified theorem is satisfied.

For the purpose of demonstrating that the left inequality in (a) is also satisfied, consider the following application of Schwarz's inequality. Let

$$(14) \qquad G(x_p, \cdots, x_k) = \int \cdots \int r_p f(x_{k+1}) \cdots f(x_{k+p-1})\, dx_{k+1} \cdots dx_{k+p-1},$$

where $f(x)$ denotes the distribution function of the variable x and the range of integration in this and subsequent integrals is from $-\infty$ to ∞. Since r_p and f are continuous non-negative functions, this integral is a positive function of the indicated variables. Then, denoting $G(x_p, \cdots, x_k)$ by G, it follows from Schwarz's inequality that

$$I = \left[\int \cdots \int r_1 f(x_1) \cdots f(x_k)\, dx_1 \cdots dx_k\right]^2$$

$$(15) \qquad = \left[\int \cdots \int \{r_1 f(x_1) \cdots f(x_k) G\}^{\frac{1}{2}} \{r_1 f(x_1) \cdots f(x_k) G^{-1}\}^{\frac{1}{2}}\, dx_1 \cdots dx_k\right]^2$$

$$\leq \int \cdots \int r_1 f(x_1) \cdots f(x_k) G\, dx_1 \cdots dx_k \int \cdots \int r_1 f(x_1) \cdots f(x_k) G^{-1}$$

$$dx_1 \cdots dx_k.$$

The two integrals of this inequality will be denoted by I_α and I_β, respectively. If the value of G given by (14) is substituted in I_α, it will be observed that

$$(16) \qquad I_\alpha = \int \cdots \int r_1 r_p f(x_1) \cdots f(x_{k+p-1})\, dx_1 \cdots dx_{k+p-1}.$$

Now I_β may be written in the form

$$I_\beta = \int \cdots \int f(x_p) \cdots f(x_k) G^{-1} \left[\int \cdots \int r_1 f(x_1) \cdots f(x_{p-1})\, dx_1 \cdots dx_{p-1}\right]$$

$$dx_p \cdots dx_k.$$

Since the x_i possess the same distribution function and r_1 is the range of the variables from x_1 to x_k, the integral in brackets is equivalent to the integral defining G in (14); hence

$$(17) \qquad I_\beta = \int \cdots \int f(x_p) \cdots f(x_k) G^{-1} G\, dx_p \cdots dx_k = 1.$$

If (16) and (17) are applied to inequality (15), they will yield the inequality

$$\left[\int \cdots \int r_1 f(x_1) \cdots f(x_k)\, dx_1 \cdots dx_k\right]^2$$

$$\leq \int \cdots \int r_1 r_p f(x_1) \cdots f(x_{k+p-1})\, dx_1 \cdots dx_{k+p-1}.$$

In statistical language, this inequality states that

$$E^2(r_1) \leq E(r_1 r_p),$$

or, what is equivalent, that

$$E^2(r_1) \leq E(r_i r_j). \tag{18}$$

If (18) is applied to (13),

$$\sigma_R^2 \geq (n-k+1)E(r_1^2) + (k-1)(2n-3k+2)E^2(r_1) + [n(1-2k) + (k-1)(3k-1)]E^2(r_1)$$

$$\geq (n-k+1)[E(r_1^2) - E^2(r_1)]$$

$$\geq \sigma_{r_1}^2(n-k+1).$$

Thus, for a fixed k the left inequality in (a) of the theorem is also satisfied, and it merely remains to be shown that condition (b) is satisfied.

For k fixed, r_i and r_{i+g} will be independently distributed provided that $g \geq k$. But if $c_3 > k$, then $c_3(n-k+1)^{c_4} > k$ for $0 < c_4 < \frac{1}{2}$ because $n - k + 1 > 1$; consequently r_i and r_{i+g} will be independently distributed for $g > c_3(n-k+1)^{c_4}$, where $0 < c_4 < \frac{1}{2}$. Thus, conditions (a) and (b) are both satisfied by R. Since $R = (n-k+1)W$, it therefore follows that

$$\frac{W - E(W)}{\sigma_W} \tag{19}$$

possesses an asymptotic normal distribution with zero mean and unit variance provided only that x possesses a continuous distribution function for which the third absolute moment exists. The existence of the third absolute moment for x insures the existence of the same moment for r_i.

If $k = 2$, W reduces to w, and therefore the validity of (11) is assured.

6. Other asymptotic distributions. The only property of the range employed in the proof of the preceding section was its positive nature; consequently the proof is applicable to moving means of other dependent statistics that are positive and possess third absolute moments.

For example, the preceding proof can be applied to δ^2 to show that δ^2 possesses an asymptotic normal distribution provided only that the sixth moment of x exists. In the study [1] of the efficiency of δ^2 for x normally distributed, no proof was given of its asymptotic property. The preceding proof could be used in studying the efficiency of δ^2, or obvious generalizations of it, as measure of variation for non-normal populations. The normality of the asymptotic distribution of the serial correlation coefficient could also be verified by means of this proof.

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CONFIDENCE LIMITS FOR THE FRACTION OF A NORMAL POPULATION WHICH LIES BETWEEN TWO GIVEN LIMITS[1]

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Summary. Let μ and σ^2 be the unknown mean and variance, respectively, of a normally distributed population on which N independent observations $x_1, \cdots, x_N$ have been made. Let L_1 and L_2, $L_1 < L_2$, and α, $0 < \alpha < 1$, be given constants. We define the following symbols:

(a) $$\gamma = (\sqrt{2\pi}\sigma)^{-1} \int_{L_1}^{L_2} \exp\left\{-\frac{1}{2}\frac{(y-\mu)^2}{\sigma^2}\right\} dy$$

(b) $$\bar{x} = N^{-1}\Sigma x_i$$

(c) $$s^2 = (N-1)^{-1}\Sigma(x_i - \bar{x})^2$$

(d) $\chi^2_{1-\alpha}$ as that number for which $P\{\chi^2 < \chi^2_{1-\alpha}\} = 1 - \alpha$ where χ^2 has $N - 1$ degrees of freedom.

(e) $$w = \sqrt{N-1}\,\frac{s}{\chi_{1-\alpha}}$$

(f) $$D = (2\pi)^{-1/2} \int_{(L_1-\bar{x})/w}^{(L_2-\bar{x})/w} \exp\left\{-\frac{1}{2}y^2\right\} dy$$

It is proved that, under restrictions stated precisely below, and before the observations are made, the probability that $D \leq \gamma$ differs from α by a number which can be made arbitrarily small by making N sufficiently large. Thus an approximate (large sample) lower confidence limit for γ is obtained. Similar methods can be applied to obtain upper and two-sided confidence limits.

A problem raised by the present paper (but not attacked here) is to investigate the rapidity of approach to α of $P\{D \leq \gamma\}$. It would perhaps be useful to obtain a series for the latter in powers of $N^{-\frac{1}{2}}$; the first term of such an expansion is obtained here.

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[1] Formula (5.1) of the present paper was given without proof by the author in July, 1945, in solution of a problem put to him by Dr. M. A. Girshick. At the time, both were members of the Statistical Research Group, formed in the Division of War Research of Columbia University under contract with the National Defense Research Committee of the Office of Scientific Research and Development. The validation of formula (5.1) in all rigor as it is given in the present paper was constructed by the author after he was no longer a member of the Statistical Research Group.

In January, 1945, Professor A. Wald, then a consultant to the Statistical Research Group, and the present author jointly submitted to the Group an unpublished memorandum (#410) entitled "Acceptance Regions Which Involve the Normal Distribution and Large Sample Sizes." While this memorandum dealt with a different problem, its ideas were logically antecedent to formula (5.1). The present author wishes to express his indebtedness to this memorandum and to his colleague Professor Wald.

1. The problem. Let μ and σ^2 be the unknown mean and variance, respectively, of a normally distributed population on which the N independent observations $x_1, x_2, \cdots, x_N$ have been made. Let L_1 and L_2 be given constants with $L_1 < L_2$. We then have that

$$\gamma = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{L_1}^{L_2} \exp\left\{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2\right\} dy$$

is the fraction of the normal population which lies between L_1 and L_2. The problem considered in this paper is to construct a lower confidence limit for the unknown γ, when N is large. An upper confidence limit or two-sided confidence limits may be constructed in a manner very similar to that described in the present paper. Since the construction of a lower limit is the problem which occurs most often in practice the discussion will be centered on it.

A lower (confidence) limit on γ with confidence coefficient α is a function $D(x_1, \cdots, x_N)$ of the observations $x_1, \cdots, x_N$ with the property that, *before* the observations are made, the probability is α that $D(x_1, \cdots, x_N) \leq \gamma$. In any specific application it is unknown whether this last inequality holds, because γ is unknown. However, one who proceeds as if this inequality were true is using a procedure which will give correct results $100\alpha\%$ of the time in the long run.

When either $L_1 = -\infty$ or $L_2 = +\infty$ the solution, by use of the non-central t distribution, is well known. For a description of the procedure and necessary tables the reader is referred to [1].

2. Acceptance regions. Let γ_0 be any value of the parameter γ. To γ_0 there correspond infinitely many couples (μ, σ) with the property that the normal distributions characterized by these couples all have a fraction γ_0 lying between L_1 and L_2; we may write this symbolically by saying that the couples (μ, σ) satisfy

$$\gamma(\mu, \sigma) = \gamma_0. \tag{2.1}$$

The construction of confidence regions is equivalent to the construction, for every γ_0, of an acceptance region $R(\gamma_0)$ in the N-dimensional Euclidean space, with the property that every normal distribution whose parameters μ and σ satisfy (2.1) assigns to $R(\gamma_0)$ the constant probability α. While this property of similarity (cf. [2]) is sufficient for the construction of confidence regions, additional properties of the acceptance regions $R(\gamma_0)$ are needed in order that the confidence region be an interval or that the upper confidence limit be always one (i.e., that the confidence limits turn out to be a lower limit only), or to insure other features deemed desirable.

It is easy to construct acceptance regions which will fulfill the condition of similarity. As an example, consider the case $N = 3$ for convenience. Let b_1, b_2, b_3 be a number triple such that $b_1 + b_2 + b_3 = 0$. Let $R(\gamma_0)$, for any given γ_0, $0 < \gamma_0 < 1$, consist of all the points x_1, x_2, x_3 which are such that the absolute value of the angle $\psi(-\pi \leq \psi \leq \pi)$ between the vector (b_1, b_2, b_3) and the vector

$(x_1 - \bar{x}, x_2 - \bar{x}, x_3 - \bar{x})$ does not lie between $\pi\alpha\gamma_0$ and $\pi + \pi\alpha(\gamma_0 - 1)$. (We define, in general, $\sum_1^N x_i = N\bar{x}$. The points (x_1, x_2, x_3) for which $x_1 = x_2 = x_3$ may be disregarded, since their probability is zero when the distribution is continuous.) One readily verifies that the probability of $R(\gamma_0)$ for any γ_0 is α, no matter what μ and σ are, and hence this is true in particular for the pairs which satisfy (2.1).

The above method of constructing acceptance regions yields confidence regions which, while they cover the unknown γ with confidence coefficient α, are not very meaningful otherwise. The fact that the probability of $R(\gamma_0)$ is α whether or not (μ, σ) satisfies (2.1) is already indicative of their lack of discrimination. Since $\bar{x}$ and s (where s is defined by

$$ns^2 = \sum_1^N (x_i - \bar{x})^2$$

and $n = N - 1$) are sufficient estimates of μ and σ, which in turn determine γ, it is clear that desirable confidence regions should be functions only of $\bar{x}$ and s. Consequently our first task must be to construct the acceptance regions $R(\gamma_0)$ in the $\bar{x}$, s plane. In the present paper we construct in the $\bar{x}$, s plane regions $R(\gamma_0)$ which have the property that their probability, under any normal distribution whose parameters satisfy (2.1), differs from the prescribed α by a quantity which is bounded in absolute value for all γ_0, in such a way that the bound approaches zero as N increases. Thus when the sample number is sizeable we can obtain confidence regions for γ which correspond to a confidence coefficient which differs little from α. Finally, the acceptance regions $R(\gamma_0)$ which we shall construct will be such that the confidence region will be always an interval, and the upper limit will always be 1, i.e., we will construct a lower confidence limit for γ.

3. Construction of regions $R(\gamma_0)$ in the $\bar{x}$, s plane. First we describe two assumptions which we shall make. It is believed that these are reasonable from the practical standpoint and are satisfied in most actual investigations where the present problem arises. Mathematically their purpose is to enable us to secure a *uniform* bound on the difference between α and the probability of $R(\gamma_0)$ (for *all* γ_0) under all couples (μ, σ) which satisfy (2.1).

ASSUMPTION 1: *There exists a positive d such that*

$$L_1 + d < \mu < L_2 - d.$$

In most practical cases where the present problem will occur γ will be larger than $\frac{1}{2}$. If the latter is the case and μ were very near either L_1 or L_2, then σ would have to be very small. In that case other methods would have to be used in the solution of the practical problem. The present paper deals with the situation, unfortunately only too common in practice, where σ is not too small. Assumption 1 puts a lower bound on σ for any given value γ_0. (The bound is a function of γ_0).

ASSUMPTION 2: *The standard deviation σ is less than a positive number C.*

In most practical problems such an upper bound can reasonably be set. Naturally, the larger d and the smaller C the more a priori information is at our disposal, the closer are our approximations and the narrower our limits. The effect of Assumptions 1 and 2 is to place a lower limit G on γ where

$$G = \gamma(L_1 + d, C) = \gamma(L_2 - d, C).$$

Let γ_0 be any positive number such that $G < \gamma_0 < 1$. For an $\bar{x}$ such that $L_1 < \bar{x} < L_2$, let $r(\bar{x}, \gamma_0)$ be the positive number such that

$$\gamma(\bar{x}, r(\bar{x}, \gamma_0)) = \gamma_0 .$$

We define $\chi^2_{1-\alpha}$ to be that number for which

$$P(\chi^2 < \chi^2_{1-\alpha}) = 1 - \alpha,$$

where χ^2 has n degrees of freedom and P is the probability of the relation in parentheses. The number $\chi^2_{1-\alpha}$ may be found in tables of the χ^2-distribution if the value of α is one of those in common use. Finally define

$$\varphi(\bar{x}, \gamma_0) = r(\bar{x}, \gamma_0) \sqrt{\frac{\chi^2_{1-\alpha}}{n}}$$

The acceptance regions $R(\gamma_0)$, $G < \gamma_0 < 1$, which we shall employ, are defined as follows for any γ_0, $G < \gamma_0 < 1$:

$$L_1 \leq \bar{x} \leq L_2$$

$$s \geq \varphi(\bar{x}, \gamma_0).$$

4. Proof that $P\{R(\gamma_0)\} \sim \alpha$. This section will be devoted to a proof of the following:

THEOREM. *Let $R(\gamma_0)$ be as defined in Section 3 for $G < \gamma_0 < 1$. Let the assumptions 1 and 2 of Section 3 be fulfilled. Then the absolute value of the difference between α and the probability of $R(\gamma_0)$ under any couple (μ, σ) which satisfies (2.1) is less than any arbitrarily small positive ϵ when N is sufficiently large, i.e., when N is sufficiently large,*

$$| P\{R(\gamma_0)\} - \alpha | < \epsilon$$

uniformly for all (μ, σ) which satisfy (2.1) with $G < \gamma_0 < 1$, and which fulfill Assumptions 1 and 2.

LEMMA 1. $\dfrac{\partial r(\bar{x}, \gamma_0)}{\partial \bar{x}}$ *exists in the open interval $L_1 < \bar{x} < L_2$.*

PROOF: We have

$$\gamma_0 = \frac{1}{\sqrt{2\pi} r(\bar{x}, \gamma_0)} \int_{L_1}^{L_2} \exp\left\{ -\frac{1}{2} \left(\frac{y - \bar{x}}{r(\bar{x}, \gamma_0)} \right)^2 \right\} dy$$

$$= \frac{1}{\sqrt{2\pi}} \int_{(L_1 - \bar{x})/r}^{(L_2 - \bar{x})/r} \exp\left\{ -\frac{1}{2} y^2 \right\} dy .$$

Differentiating with respect to $\bar{x}$ we obtain, since $r > 0$,

$$e^{-R^2/2}\left(1 + R\frac{\partial r}{\partial \bar{x}}\right) = e^{-T^2/2}\left(1 + T\frac{\partial r}{\partial \bar{x}}\right)$$

with

$$R = \frac{L_2 - \bar{x}}{r} \qquad T = \frac{L_1 - \bar{x}}{r}.$$

Hence

$$\frac{\partial r}{\partial \bar{x}} = -\left(\frac{e^{-R^2/2} - e^{-T^2/2}}{Re^{-R^2/2} - Te^{-T^2/2}}\right). \tag{4.1}$$

Since $R > 0$ and $T < 0$ within the open interval $L_1 < \bar{x} < L_2$, it follows that $\frac{\partial r}{\partial \bar{x}}$ exists in the entire open interval.

LEMMA 2. *In the open interval* $L_1 < \bar{x} < L_2$,

$$\frac{\partial P\{s \geq \varphi(\bar{x}, \gamma_0)\}}{\partial \bar{x}}$$

exists.

PROOF: We have, with k a suitable constant,

$$P = k\int_{\sqrt{n}\varphi/\sigma}^{\infty} y^{n-1}e^{-y^2/2}\,dy = k\int_{(r(\bar{x},\gamma_0)/\sigma)\chi_{1-\alpha}}^{\infty} y^{n-1}e^{-y^2/2}\,dy.$$

Hence

$$\frac{\partial P}{\partial \bar{x}} = \frac{-k\chi_{1-\alpha}}{\sigma}\frac{\partial r}{\partial \bar{x}}\left(\frac{r\cdot\chi_{1-\alpha}}{\sigma}\right)^{n-1}\exp\left(\frac{-r^2\chi_{1-\alpha}^2}{2\sigma^2}\right). \tag{4.2}$$

LEMMA 3. *Let* δ *be any arbitrarily small positive number. The function* $\left|\frac{\partial P}{\partial \bar{x}}\right|$ *of* $\bar{x}$ *and* γ_0 *is bounded for* $L_1 + \delta \leq \bar{x} \leq L_2 - \delta$, $G < \gamma_0 < 1$.

PROOF: From (4.1) we have

$$\left|\frac{\partial r}{\partial \bar{x}}\right| < \frac{e^{-R^2/2} + e^{-T^2/2}}{Re^{-R^2/2} - Te^{-T^2/2}} \leq \max.\left(\frac{1}{R}, \frac{-1}{T}\right) = \max.\left(\frac{r}{L_2 - \bar{x}}, \frac{r}{\bar{x} - L_1}\right) \leq \frac{r}{\delta}.$$

Therefore from (4.2) we have that $\left|\frac{\partial P}{\partial \bar{x}}\right|$ is less than a constant multiplied by $\left(\frac{r}{\sigma}\right)^n \exp\left(\frac{-r^2\chi_{1-\alpha}^2}{2\sigma^2}\right)$ and is therefore bounded.

PROOF OF THE THEOREM: From Lemma 3 and the Theorem of the Mean it follows that, in the closed interval

$$L_1 + \frac{d}{2} \leq \bar{x} \leq L_2 - \frac{d}{2},$$

the function $P\{s \geq \varphi(\bar{x}, \gamma_0)\}$ is uniformly continuous in $\bar{x}$ uniformly for all (μ, σ) which satisfy (2.1) with $G < \gamma_0 < 1$. Hence for every positive ϵ_1 there

exists a positive $\eta < \frac{d}{2}$ such that $| l_1 - l_2 | < \eta$,

$$L_1 + \frac{d}{2} \leq l_1, l_2 \leq L_2 - \frac{d}{2},$$

implies

$$| P\{s \geq \varphi(l_1, \gamma_0)\} - P\{s \geq \varphi(l_2, \gamma_0)\} | < \epsilon_1 .$$

For fixed arbitrary $\epsilon_2 > 0$ we have, when N is sufficiently large,

$$P\{ | \bar{x} - \mu | < \eta\} > 1 - \epsilon_2 ,$$

from Assumption 2 and the stochastic convergence of $\bar{x}$. Now

$$P\{s \geq \varphi(\mu, \gamma_0)\} = \alpha.$$

Hence, when N is sufficiently large,

$$| P\{R(\gamma_0)\} - \alpha | \leq \epsilon_1(1 - \epsilon_2) + \epsilon_2 \leq \epsilon_1 + \epsilon_2 .$$

Since ϵ_1 and ϵ_2 are arbitrarily small, this proves the desired result.

5. Construction of large sample confidence regions. The acceptance regions $R(\gamma_0)$ whose size never differs from α by more than a uniform bound which approaches zero as N increases, readily yield a lower confidence limit for γ (within the approximation involved). The confidence region consists of all the γ_0 for which $R(\gamma_0)$ contains the observed $\bar{x}$, s. Our acceptance regions $R(\gamma_0)$ are so constructed that, if $\gamma_1 < \gamma_2$, $R(\gamma_1)$ is entirely contained within $R(\gamma_2)$. Hence the confidence region is an interval, one end of which is always unity, as was desired. The rule for constructing the lower confidence limit D is, therefore, as follows:

a) if $\bar{x} < L_1$ or $\bar{x} > L_2$, then $D = G$

b) if $L_1 \leq \bar{x} \leq L_2$, then

$$D = \frac{1}{\sqrt{2\pi}} \int_{(L_1 - \bar{x})/w}^{(L_2 - \bar{x})/w} \exp\{-\tfrac{1}{2}y^2\}\, dy \tag{5.1}$$

where

$$w = \sqrt{N - 1} \cdot \frac{s}{\chi_{1-\alpha}}.$$

(The value of D may be found in a table of the normal distribution. It is easy to see that $s = \varphi(\bar{x}, D)$, i.e., D is the smallest value of γ_0 for which $\bar{x}$, s will still lie in $R(\gamma_0)$).

If the statement $D \leq \gamma$ is made in a large number of cases, where the assumptions are fulfilled and the sample size is large, the proportion of correct statements will be close to α.

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NOTES

This section is devoted to brief research and expository articles on methodology and other short items.

ON SEQUENTIAL BINOMIAL ESTIMATION

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The present note, written after a reading of the very interesting paper by Girshick, Mosteller, and Savage [1], is for the purpose of adding a few remarks in the nature of a supplement. For the sake of brevity the notation and terminology of [1] are adopted in toto.

Theorem 1 below generalizes Theorem 1 of [1]. In Theorem 2′ we formulate explicitly the fact which lies at the basis of the GSM method of estimation. Parts of the proofs of Theorems 3 and 4 of [1] are simply proofs of special cases of this (e.g., equation (2) of [1]). We then use this fact repeatedly in proving Theorem 3, which states that the Girshick-Mosteller-Savage estimate is the only proper unbiased estimate for sequential tests defined by regions which we shall call doubly simple.

A doubly simple region is defined precisely below. Intuitively we may describe such a region as the one between two curves $y = f_1(x)$ and $x = f_2(y)$, where $f_1(x)$ is defined and monotonically non-decreasing for all non-negative x, $f_2(y)$ is defined and monotonically non-decreasing for all non-negative y, $f_1(0) > 0$, $f_2(0) > 0$. If the two curves intersect, the region is finite, and the values of the functions f_1 and f_2 beyond the point of intersection are of no interest. This description is of course purely heuristic, because in actual fact only integral values of the variables come into play, and intersection of the curves, for example, is not needed to make the region finite. Since the question of finite regions is completely settled by [1], Theorem 7, only non-finite regions remain to be discussed, and the precise definition given below is such as to imply that the region is not finite. It seems to the present writer that at least many of the non-finite sequential tests which may be developed for meaningful statistical problems will require doubly simple regions. The Wald sequential binomial test [2] defines such a region, which also falls within the scope of Theorem 6 of [1]. It is easy to see that there exist closed regions which are doubly simple and do not satisfy the conditions of this theorem.

By a "proper" estimate $p(\alpha)$ we shall mean an estimate such that $0 \leq p(\alpha) \leq 1$ for every α. It is difficult to see how any estimate which is not proper can make much sense.

Theorem 1. *A sufficient condition that a region R be closed is that* $\liminf_{n\to\infty} \frac{A(n)}{\sqrt{n}} < \infty$, *where $A(n)$ is the number of accessible points of index n.*

PROOF: The hypothesis of the theorem implies that there exist a positive number H and an increasing sequence of positive integers $n_1, n_2, n_3, \cdots$, with the following properties:

a) $n_{i+1} > 2n_i$ $(i = 1, 2, \cdots$ ad inf.)

b) $A(n_i) < H\sqrt{n_i}$.

For n_i sufficiently large, the conditional probability of reaching the accessible points on $x + y = n_{i+1}$, when an accessible point on $x + y = n_i$ has been reached, is $< K < 1$ by the normal approximation to the binomial distibution, where K is constant (and depends on H). Hence the probability of passing through accessible points on all members of the set $x + y = n_i$ $(i = 1, 2, \cdots, L)$ approaches zero as $L \to \infty$, so that the region is closed.

THEOREM 2. *Let R be any region, B its boundary, and $t = (a, b)$; any accessible point in R. Let $l_t(\alpha)$ be the number of paths from t to $(x, y) = \alpha \epsilon B$. Let $Q(t)$ be the conditional probability that a path, which has reached t, will reach the boundary B. Then*

$$\sum_{\alpha \epsilon B} l_t(\alpha) p^y q^x = Q(t) p^b q^a.$$

THEOREM 2'. (Corollary to Theorem 2)

If R is closed, then

$$\sum_{\alpha \epsilon B} l_t(\alpha) p^y q^x = p^b q^a. \tag{1}$$

PROOF: Let $k(t)$ be the number of paths in R from the origin to t. The probability of reaching $\alpha \epsilon B$ by a path which passes through t is $k(t) l_t(\alpha) p^y q^x$. The probability of reaching t from the origin is $k(t) p^b q^a$, and hence the probability of reaching the boundary via t is $Q(t) k(t) p^b q^a$. From this the desired result follows.

We now define a doubly simple region. The boundary of the region consists of the two infinite sequences of points

$$(0, a_0), (1, a_1), (2, a_2), \cdots$$

and

$$(b_0, 0), (b_1, 1), (b_2, 2), \cdots$$

where $a_0, a_1, a_2, \cdots$ and $b_0, b_1, b_2, \cdots$ are two infinite non-decreasing sequences of positive integers. The accessible points of the region are all points which can be reached by a path from the origin which does not contain a boundary point. (It is to be noted that since a boundary point is, by definition, a point not in the region which can be reached by a path in the region, the above definition implies that a doubly simple region is not finite. The reason for making this so has been given above.)

THEOREM 3. Let R be a closed doubly simple region. Then $\hat{p}(\alpha)$ is the unique proper unbiased estimate of p.

PROOF: Suppose there were two proper unbiased estimates $p_1(\alpha)$ and $p_2(\alpha)$. Writing $m(\alpha) = p_1(\alpha) - p_2(\alpha)$, we would have

$$\sum_{\alpha \in B} m(\alpha)k(\alpha)p^y q^x = 0 \tag{2}$$

with

$$| m(\alpha) | \leq 1 \tag{3}$$

First we prove

LEMMA 1. *If* $a_0 > 1$, *then* $m(b_0, 0) = 0$.

PROOF: Let $k^*(\alpha)$ denote the number of paths in R from the point (0, 1) to the boundary point α. For all points $\alpha \in B$ except $(b_0, 0)$ we have

$$b_0 k^*(\alpha) \geq k(\alpha). \tag{4}$$

From (1), (2), (3), and (4) we have, since $k(b_0, 0) = 1$,

$$\begin{aligned} | m(b_0, 0) | q^{b_0} &= \Big| \sum_{\alpha \in B, \alpha \neq (b_0,0)} m(\alpha)k(\alpha)p^y q^x \Big| \\ &\leq \sum_{\alpha \in B, \alpha \neq (b_0,0)} k(\alpha)p^y q^x \leq b_0 \sum_{\alpha \in B} k^*(\alpha)p^y q^x = b_0 p. \end{aligned} \tag{5}$$

Now as $p \to 0$, the left member of the inequality (5) approaches $| m(b_0, 0) |$, and the right member approaches zero. This proves Lemma 1.

LEMMA 2. *For every* $z < a_0 - 1$, $m(b_z, z) = 0$.

PROOF: In view of Lemma 1 it is sufficient to prove the following:

If $Z \leq a_0 - 2$, and if $m(b_z, z) = 0$ for $z = 0, 1, \cdots, Z - 1$, then $m(b_Z, Z) = 0$. Let $k_{Z+1}(\alpha)$ denote the number of paths in R from $(0, Z + 1)$ to the boundary point α. For any point $\alpha \in B$ whose ordinate is $\geq Z + 1$ we have

$$b_0 b_1 \cdots b_Z k_{Z+1}(\alpha) \geq k(\alpha). \tag{6}$$

From (1), (2), (3), and (6) we have

$$\begin{aligned} | m(b_Z, Z) | k(b_Z, Z)p^Z q^{b_Z} &= | \Sigma m(\alpha)k(\alpha)p^y q^x | \leq \Sigma k(\alpha)p^y q^x \\ &\leq b_0 b_1 \cdots b_Z \Sigma k_{Z+1}(\alpha)p^y q^x = b_0 b_1 \cdots b_Z p^{Z+1} \end{aligned} \tag{7}$$

where the summations take place over all boundary points whose ordinates are $\geq Z + 1$. Hence

$$| m(b_Z, Z) | k(b_Z, Z)q^{b_Z} \leq b_0 b_1 \cdots b_Z p.$$

and letting $p \to 0$ we obtain the desired result.

LEMMA 3. $m(b_{a_0-1}, a_0 - 1) = 0$.

PROOF: Let s be the smallest integer such that (s, a_0) is an accessible point.

We proceed as in Lemma 2, with (s, a_0) playing the role of $(0, Z + 1)$, and eventually obtain the following inequality:

$$\begin{aligned} | m(b_{a_0-1}, a_0 - 1) | k(b_{a_0-1}, a_0 - 1)p^{a_0-1} q^{b_{a_0-1}} &= | \textstyle\sum_0 m(\alpha)k(\alpha)p^y q^x | \\ &\leq p^{a_0} \left(\sum_{i=0}^{s-1} k(i, a_0)q^i + b_0 b_1 \cdots b_{a_0-1} q^s \right), \end{aligned} \tag{8}$$

where Σ_0 denotes summation over all boundary points with ordinate $\geq a_0$. The desired result follows.

LEMMA 4. *Let $h(\geq a_0)$ be the smallest ordinate for which at least one boundary point (w^*, h) exists such that $m(w^*, h) \neq 0$ (If no such h exists the theorem is proved). Of all such points let w be the one with the smallest abscissa. Then the point (w, h) is a member of the sequence*

$$(0, a_0)\ (1, a_1),\ (2, a_2),\ \cdots$$

PROOF: If the lemma is not true, then for all boundary points α with ordinate h, $m(\alpha) = 0$, except that $m(b_h, h) \neq 0$. Let W be that accessible point of R whose ordinate is $h + 1$ and whose abscissa v is a minimum. Let $k_w(\alpha)$ be the number of paths in R from W to the boundary point α. For boundary points α accessible from W we have

$$b_0 b_1 \cdots b_h k_W(\alpha) \geq k(\alpha). \tag{9}$$

From (1), (2), (3), and (9) we have

$$\begin{aligned} |\, m(b_h, h)\,|\, k(b_h, h) p^h q^{b_h} = |\, \Sigma_1 (m(\alpha) k(\alpha) p^y q^x \,| \leq \Sigma_2 k(\alpha) p^{h+1} q^x \\ + b_0 b_1 \cdots b_h p^{h+1} q^v = K^* p^{h+1}, \end{aligned} \tag{10}$$

where:

a) Σ_1 denotes summation over all $\alpha \in B$ for which $y > h$

b) Σ_2 denotes summation over all boundary points α of ordinate $h + 1$ and abscissa $< v$.

c) K^* denotes a constant.

From this it easily follows that $m(b_h, h) = 0$, in contradiction to the definition of h. This proves Lemma 4.

PROOF OF THEOREM 3: Let (w, h) be as defined in the statement of Lemma 4. From Lemma 4 it follows that, if any other boundary points with abscissa w exist, they must be members of the sequence $(b_0, 0)$, $(b_1, 1)$, $(b_2, 2)$, $\cdots$ and hence their ordinates are $< h$. From the definition of (w, h) and from Lemma 4 it follows that for any $\alpha \in B$ whose abscissa is $< w$, $m(\alpha) = 0$.

Now in the proofs of Lemmas 1–4 the roles of x and y are not symmetrical. However, symmetry of course exists, and analogous lemmas follow. In particular, the analogue to Lemma 4 has as a consequence that, since w is the smallest abscissa such that $m(\alpha) = 0$ when abscissa of $\alpha < w$, and $m(w, h) \neq 0$, there exists a boundary point (w, h'), such that $m(w, h') \neq 0$ and (w, h') is a member of $(b_0, 0)$, $(b_1, 1)$, $(b_2, 2)$, ... Then $h' < h$. But this contradicts the definition of h and proves the theorem.

It is easy to see that, if the boundary points of a closed region constitute a single "curve" instead of two "curves" as in a doubly simple region, the estimate $\hat{p}(\alpha)$ will be the only proper unbiased estimate of p.

It is interesting to consider some of the consequences of Theorem 3 for all unbiased estimates (not necessarily proper) for doubly simple regions. An

examination of the proof of Theorem 3 shows that it would go through with little change if equation (3) were replaced by the requirement that $|m(\alpha)|$ be bounded. We therefore obtain the following result: If for a doubly simple region there exists an unbiased estimate $p(\alpha)$ of p, not identically equal to $\hat{p}(\alpha)$, then not only is $p(\alpha)$ not proper, but also, no matter how large M, there exists a boundary point α such that $|p(\alpha)| > M$. The uselessness of such an estimate is manifest.

The author is of the opinion that freedom from bias is not necessarily an indispensable characteristic of an optimum estimate. In general there is no reason for requiring the first moment of the estimate rather than any other moment to be the unknown parameter. The justification in any particular case must be based on special conditions of the problem.

The author is indebted to Mr. Howard Levene for reading the present paper and making valuable suggestions.

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DIFFERENTIATION UNDER THE EXPECTATION SIGN IN THE FUNDAMENTAL IDENTITY OF SEQUENTIAL ANALYSIS

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1. Introduction. Let $\{z_\alpha\}$ ($\alpha = 1, 2, \cdots$, ad inf.) be a sequence of random variables which are independently distributed with identical distributions. Let a be a positive, and b a negative constant. For each positive integral value m, let Z_m denote the sum $z_1 + \cdots + z_m$. Denote by n the smallest integral value for which Z_n does not lie in the open interval (b, a). For any random variable u, let the symbol $E(u)$ denote the expected value of u. The following identity, which plays a fundamental role in sequential analysis, has been proved in [1].

$$E[e^{Z_n t}\varphi(t)^{-n}] = 1, \tag{1.1}$$

where

$$\varphi(t) = E(e^{zt}) \tag{1.2}$$

and the distribution of z is equal to the common distribution of $z_1, z_2, \cdots$, etc. Identity (1.1) holds for all points t in the complex plane for which $\varphi(t)$ exists and $|\varphi(t)| \geq 1$.

The purpose of this paper is to formulate conditions under which we may differentiate (1.1) with respect to t under the expectation sign. This is of interest, since various results in sequential analysis can easily be established by differentiating (1.1) under the expectation sign. For example, the formula for $E(n)$ can immediately be obtained by differentiating (1.1) at $t = 0$. The derivative of $e^{Z_n t}\varphi(t)^{-n}$ at $t = 0$ is given by

$$Z_n - \frac{\varphi'(0)}{\varphi(0)} n = Z_n - E(z)n \tag{1.3}$$

where $\varphi'(t)$ denotes the derivative of $\varphi(t)$. Hence, if we may differentiate (1.1) under the expectation sign, we obtain the basic formula

$$E(Z_n) = E(z)E(n). \tag{1.4}$$

If $E(z) \neq 0$, the above equation has been used [2] to derive lower and upper limits for $E(n)$. If, however, $E(z) = 0$, formula (1.4) is of little value. It will be shown in section 3 that

$$E(n) = \frac{E(Z_n^2)}{E(Z^2)} \quad \text{when} \quad E(z) = 0. \tag{1.5}$$

This result is obtained, as will be seen in section 3, by differentiating identity (1.1) twice at $t = 0$.

2. A sufficient condition for the differentiability of (1.1) under the expectation sign. In what follows, the parameter t in (1.1) will be restricted to real values, even if this is not stated explicitly. For any random variable u and any relation R, the symbol $E(u \mid R)$ will denote the conditional expected value of u under the restriction that R holds. In this section we shall establish the following theorem.

THEOREM 2.1. *If $\varphi(t)$ exists for all real values t, identity* (1.1) *may be differentiated under the expectation sign any number of times with respect to t at any value t in the domain $\varphi(t) \geq 1$.*

PROOF: First we shall derive an upper bound for $E(e^{tZ_n} \mid n = m)$ for any given integral value m. Consider the case when $t > 0$. Then

$$E(e^{tZ_n} \mid n = m) \leq E(e^{tZ_n} \mid Z_n \geq a,\ n = m) \qquad (t > 0). \tag{2.1}$$

Clearly,

$$E(e^{tZ_n} \mid Z_n \geq a, n = m, e^{tZ_{n-1}} = \rho e^{at}) = e^{at}\rho E\left(e^{zt} \mid e^{zt} \geq \frac{1}{\rho}\right). \tag{2.2}$$

Let $l(t)$ denote the least upper bound of the expression

$$\rho E\left(e^{zt} \mid e^{zt} \geq \frac{1}{\rho}\right) \tag{2.3}$$

with respect to ρ over the interval $(e^{-(a-b)|t|}, 1)$. The existence of $\varphi(t)$ implies[8] that $l(t)$ is finite. It follows from (2.1) and (2.2) that

$$E(e^{tZ_n} \mid n = m) \leq e^{at} l(t) \qquad (t > 0) \tag{2.4}$$

and, therefore, also

$$E(e^{tZ_n}) \leq e^{at} l(t) \qquad (t > 0). \tag{2.5}$$

If $t < 0$, one can show in a similar way that

$$E(e^{tZ_n} \mid n = m) \leq e^{bt} l(t) \qquad (t < 0) \tag{2.6}$$

and

$$E(e^{tZ_n}) \leq e^{bt} l(t) \qquad (t < 0). \tag{2.7}$$

To prove Theorem 2.1, it is sufficient to show that the following two propositions hold.[1]

PROPOSITION 2.1. *All derivatives of $e^{Z_n t}\varphi(t)^{-n}$ with respect to t exist in the domain $\varphi(t) \geq 1$.*

PROPOSITION 2.2. *For any positive integral value r and for any finite interval I in which $\varphi(t) \geq 1$, it is possible to find a function $D(Z_n, n)$ such that*

$$D(Z_n, n) \geq \left| \frac{d^r}{dt^r} [e^{Z_n t} \varphi(t)^{-n}] \right| \tag{2.8}$$

for all values t in I and

$$E[D(Z_n, n)] < \infty. \tag{2.9}$$

Proposition 2.1 is clearly true, if all derivatives of $\varphi(t)$ exist. The existence of these derivatives follows from the existence of $\varphi(t)$ for all values t.

Since $\frac{d^r}{dt^r} e^{Z_n t}\varphi(t)^{-n}$ is equal to the sum of a finite number of terms of the type $Z_n^{r_1} n^{r_2} e^{Z_n t}\varphi(t)^{-n}$, Proposition 2.2 is proved if we can show that for any given integral values r_1 and r_2 there exists a function $D_{r_1 r_2}(Z_n, n)$ such that

$$D_{r_1 r_2}(Z_n, n) \geq | Z_n^{r_1} n^{r_2} e^{Z_n t}\varphi(t)^{-n} | \tag{2.10}$$

for all t in I and

$$E[D_{r_1 r_2}(Z_n, n)] < \infty. \tag{2.11}$$

Clearly, since $\varphi(t) \geq 1$ in I,

$$| Z_n^{r_1 r_2} e^{Z_n t}\varphi(t)^{-n} | \leq | Z_n^{r_1} | n^{r_2} e^{|Z_n| t_0} \tag{2.12}$$

where t_0 is an upper bound of $|t|$ in I. Let t_1 be a value $> t_0$. Then for a properly chosen constant C we have

$$| Z_n^{r_1} | e^{|Z_n| t_0} < C e^{|Z_n| t_1}. \tag{2.13}$$

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[1] See, for example, E. J. McShane, *Integration*, Princeton University Press (1944), p. 216, 217 and 276.

Hence, it follows from (2.12) and (2.13) that

$$|Z_n^{r_1} n^{r_2} e^{Z_n t} \varphi(t)^{-n}| \leq C n^{r_2} e^{|Z_n| t_1} \leq C n^{r_2} (e^{Z_n t_1} + e^{-Z_n t_1}) \tag{2.14}$$

for all t in I.

We put

$$D_{r_1 r_2}(Z_n, n) = C n^{r_2} (e^{Z_n t_1} + e^{-Z_n t_1}). \tag{2.15}$$

We have

$$E[D_{r_1 r_2}(Z_n, n)] = C \sum_{m=1}^{\infty} p_m m^{r_2} [E(e^{Z_n t_1} \mid n = m) + E(e^{-Z_n t_1} \mid n = m)] \tag{2.16}$$

where p_m denotes the probability that $n = m$.

Hence, because of (2.4) and (2.6), we obtain

$$E[D_{r_1 r_2}(Z_n, n)] \leq C(e^{a t_1} l(t_1) + e^{-b t_1} l(-t_1))[\Sigma p_m m^{r_2}] = \\ = C[e^{a t_1} l(t_1) + e^{-b t_1} l(-t_1)] E(n^{r_2}). \tag{2.17}$$

Since all moments of n are finite,[2] Proposition 2.2 is proved. This completes the proof of Theorem 2.1.

3. The expected value of n when $E(z) = 0$. It will be shown in this section that

$$E(n) = \frac{E(Z_n^2)}{E(z^2)} \quad \text{when} \quad E(z) = 0, \tag{3.1}$$

if identity (1.1) can be differentiated twice under the expectation sign at $t = 0$. The second derivative of $e^{t Z_n} \varphi(t)^{-n}$ with respect to t is given by

$$\left\{ \left[Z_n - n \frac{\varphi'(t)}{\varphi(t)} \right]^2 - n \frac{\varphi''(t)\varphi(t) - [\varphi'(t)]^2}{[\varphi(t)]^2} \right\} e^{Z_n t} \varphi(t)^{-n} \tag{3.2}$$

where $\varphi'(t)$ denotes the first, and $\varphi''(t)$ the second derivative of $\varphi(t)$.

Since $\varphi(0) = 1$, $\varphi'(0) = E(z) = 0$ and $\varphi''(0) = E(z^2)$, putting $t = 0$, expression (3.2) becomes

$$Z_n^2 - n\varphi''(0) = Z_n^2 - nE(z^2) \tag{3.3}$$

Hence, if (1.1) may be differentiated twice under the expectation sign at $t = 0$, we obtain

$$E[Z_n^2 - nE(z^2)] = 0 \tag{3.4}$$

from which (3.1) follows.

An approximate value of $E(n)$ can be obtained from (3.1) by neglecting the excess of Z_n over the boundaries. Then Z_n can take only the values a and b. Hence

$$E(Z_n^2) \sim a^2 P(Z_n \geq a) + b^2 P(Z_n \leq b) \tag{3.5}$$

where the sign $\sim$ denotes approximate equality.

[2] See the paper by C. Stein, "A note on cumulative suns," in this issue of the *Annals of Mathematical Statistics*.

It was shown in [1] (equation 28) that neglecting the excess of Z_n over the boundaries, the approximation formula

$$P(Z_n \geq a) \sim \frac{1 - e^{bh}}{e^{ah} - e^{bh}} \tag{3.6}$$

holds, where h is the non-zero root of the equation $\varphi(t) = 1$. This formula was derived there under the assumption that $E(z) \neq 0$. If $E(z)$ approaches zero, $h \to 0$ and the right hand member of (3.6) converges to $\frac{-b}{a-b}$.

Putting $P(Z_n \geq a) = \frac{-b}{a-b}$ and $P(Z_n \leq b) = 1 - \frac{-b}{a-b} = \frac{a}{a-b}$, we obtain from (3.5)

$$E(Z_n^2) \sim a^2\left(\frac{-b}{a-b}\right) + b^2\frac{a}{a-b} = -ab. \tag{3.7}$$

Hence[3]

$$E(n) \sim \frac{-ab}{E(z^2)}. \tag{3.8}$$

Limits for $E(n)$ can be obtained by deriving limits for $E(Z_n^2)$. Let r be a non-negative real variable. One can verify that

$$a^2 \leqq E(Z_n^2 \mid Z_n \geq a) \leqq \underset{0<r<a-b}{\text{l.u.b.}}\ E[(a - r + z)^2 \mid z \geq r] \tag{3.9}$$

and

$$b^2 \leqq E(Z_n^2 \mid Z_n \leqq b) \leq \underset{0<r<a-b}{\text{l.u.b.}}\ E[(b + r + z)^2 \mid z + r \leq 0]. \tag{3.10}$$

We have

$$E(Z_n^2) = P(Z_n \geq a)E(Z_n^2 \mid Z_n \geq a) + P(Z_n \leq b)E(Z_n^2 \mid Z_n \leq b). \tag{3.11}$$

Limits for $E(Z_n^2)$ can be obtained by replacing the conditional expected values in the right hand member of (3.11) by their limits given in (3.9) and (3.10).

REFERENCES

[1] A. WALD, "On cumulative sums of random variables," *Annals of Math. Stat.*, Vol. 15 (1944), pp. 285, 287.

[2] A. WALD, "Sequential tests of statistical hypotheses," *Annals of Math. Stat.*, Vol. 16 (1946), p. 143.

[3] This approximation formula was obtained also by W. A. Wallis independently of the author. It is included in the publication of the Statistical Research Group of Columbia Univ., *Techniques of Statistical Analysis*, Chapter 17, Section 7.2, McGraw Hill, New York (1946).

A NOTE ON CUMULATIVE SUMS

By Charles Stein

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Let $\{Z_i\}$ be a denumerable sequence of identical independent real-valued random variables. Two constants $a > 0 > b$ are chosen and the random variable n defined as the smallest integer for which one of the inequalities $\sum_1^n Z_i \geq a$ $\sum_1^n Z_i \leq b$ holds. For any events E_1 and E_2, $P\{E_1\}$ will denote the probability of the event E_1 and $P\{E_1 \mid E_2\}$ the conditional probability of the event E_1 given that E_2 has occurred.

It will be shown that there exists $t_0 > 0$ such that the moment generating function, Ee^{nt} exists for any complex number t whose real part is less than or equal to t_0, and as an immediate consequence that n has finite moments of all orders.

If d is any constant satisfying $b < d < a$, then, for fixed m,

$$P\left\{b < \sum_1^m Z_i + d < a\right\} \leq P\left\{\left|\sum_1^m Z_i\right| < c\right\} \tag{1}$$

where $c = |a| + |b|$. We exclude the case $P\{Z_i = 0\} = 1$. Then there exists $\epsilon > 0$ such that either

$$\delta_1 = P\{Z_i \geq \epsilon\} > 0 \quad \text{or} \quad \delta_2 = P\{Z_i \leq -\epsilon\} > 0.$$

Taking, for example, the former alternative with $m_1 = \left[\frac{c}{\epsilon}\right] + 1$,

$$P\left\{\left|\sum_1^{m_1} Z_i\right| \geq c\right\} \geq P\left\{Z_i \geq \epsilon \quad \text{for} \quad i = 1, \cdots, m_1\right\} = \delta_1^{m_1} > 0 \tag{2}$$

where $[w]$ denotes the largest integer less than or equal to w. For any poitive integer k,

$$\frac{P\{n > km_1\}}{P\{n > (k-1)m_1\}} = P\{n > km_1 \mid n > (k-1)m_1\}$$

$$\leq P\left\{b < \sum_1^{km_1} Z_i < a \,\middle|\, b < \sum_1^{s} Z_i < a \quad \text{for} \quad s = 1, \cdots, (k-1)m_1\right\}$$

since $n > km_1$ implies $b < \sum_1^{km_1} Z_i < a$.

But $\sum_1^{km_1} Z_i = \sum_1^{(k-1)m_1} Z_i + \sum_{(k-1)m_1+1}^{km_1} Z_i$ and the second sum on the right hand side is independent of all terms in the first sum.

Thus the distribution of $\sum_1^{km_1} Z_i$ given $\sum_1^{s} Z_i$ for $s = 1, \cdots, (k-1)m_1$ depends only on $\sum_1^{(k-1)m_1} Z_i$ so that

$$\frac{P\{n > km_1\}}{P\{n > (k-1)m_1\}} \leq P\left\{b < \sum_{(k-1)m_1+1}^{km_1} Z_i + \sum_1^{(k-1)m_1} Z_i < a \,\middle|\, b < \sum_1^{(k-1)m_1} Z_i < a\right\} \tag{3}$$

$$\leq P\left\{\left|\sum_{(k-1)m_1+1}^{km_1} Z_i\right| < c\right\} \leq 1 - \delta_1^{m_1} \quad \text{by (1) and (2).}$$

Consequently, by induction on k,

$$P\{n > m\} \leq P\left\{n > \left[\frac{m}{m_1}\right] m_1\right\} \leq (1 - \delta_1^{m_1})^{[\frac{m}{m_1}]}. \tag{4}$$

Let t_0 be any positive number less than $-\frac{1}{m_1}\log(1 - \delta_1^{m_1})$.

Then

$$\begin{aligned} Ee^{nt_0} &= \sum_{m=1}^{\infty} e^{mt_0} P\{n = m\} \\ &\leq \sum_{k=1}^{\infty} e^{km_1 t_0} P\{(k-1)m_1 < n \leq km_1\} \\ &\leq \sum_{k=1}^{\infty} e^{km_1 t_0} P\{n > (k-1)m_1\} \qquad (5) \\ &\leq \sum_{k=1}^{\infty} e^{km_1 t_0}(1 - \delta_1^{m_1})^{k-1} \\ &= \frac{1}{1 - \delta_1^{m_1}} \sum_{k=1}^{\infty} \{e^{m_1 t_0}(1 - \delta_1^{m_1})\}^k. \end{aligned}$$

But this is a geometric series with decreasing terms, and is consequently convergent. Thus for any t whose real part $R(t) \leq t_0$, the moment generating function Ee^{nt} exists. Since, for all positive l, $m^l < e^{mt_0}$ for sufficiently large m, n has finite moments of all orders.

ABSTRACTS OF PAPERS

Presented on August 21, 1946, at the Cornell meeting of the Institute

1. **A Test of Randomness in Two Dimensions.** HOWARD LEVENE, Columbia University.

A square of side N is divided into N^2 unit cells, and each cell takes on the characteristics A or B with probabilities p and $q = 1 - p$ respectively, independently of the other cells. A cell is an "upper left corner" if it is A and the cell above and cell to the left are not A. Let V_1 be the total number of upper left corners and let V_2, V_3, V_4 be the number of similarly defined upper right, lower right, and lower left corners respectively. Let $V = (V_1 + V_2 + V_3 + V_4)/4$. It is proved that V is normally distributed in the limit with $E(V) = p(Nq + p)^2$ and $\sigma^2(V) \sim N^2pq^2(4 - 20p + 45p^2 - 27p^3)/4$. The conditional limit distribution of V when p is estimated from the data, and the limit distribution of a related quadratic form are also obtained. These statistics are in a sense a generalization of the run statistics used for testing randomness in one dimension.

2. **Asymptotic Distribution of Moments from a System of Linear Stochastic Difference Equations.** HERMAN RUBIN, Cowles Commission for Research in Economics.

Let $\sum_{\tau=0}^{\infty} B_\tau y'_{t-\tau} + \Gamma z'_t = u'_t$, $(t = 1, 2, \cdots)$, be a complete system of linear stochastic difference equations determining y_{ti} (the coordinates of y_t), $t > 0$, in terms of y_{ti}, $t \leq 0$, and z_{tk} (the coordinates of z_t), which are assumed to be fixed variates, and the random variables u_{ti} (the coordinates of u_t). Such a system is called a stable if for every bounded set of fixed variates, and $E(u'_t u_t)$ uniformly bounded, $E(y'_t y_t)$ is uniformly bounded. This condition is shown to be equivalent to $\sum |h_{ij\tau}|$ finite, where $y'_t = \sum_{\tau=0}^{\infty} H_\tau(u'_{t-\tau} - \Gamma z'_{t-\tau}) + \sum_{\nu=0}^{\infty} J_{t,\nu} y'_{-\nu}$ is the solution of the above difference equation. Let Q_t be an infinite quadratic form in $y_{t-\tau,i}$ and $z_{t-\nu,k}$ $(\tau, \nu = 0, 1, \cdots)$ with coefficients depending only on i, k, τ, and ν. Such a quadratic form is called convergent if the sum of the absolute values of the coefficients is finite. It is shown under fairly general conditions that the mean of a convergent quadratic form is asymptotically normally distributed with variance $0\left(\frac{1}{T}\right)$.

3. **Conditional Expectation and Unbiased Sequential Estimation.** DAVID BLACKWELL, Howard University.

It is shown that $E[f(x_\alpha)E_\alpha y] = E(fy)$ whenever $E(fy)$ is finite, and that $\sigma^2(E_\alpha y) \leq \sigma^2(y)$, with equality holding only if $E_\alpha y = y$, where $E_\alpha y$ denotes the conditional expectation of y with respect to the family of chance variables x_α. These results imply that whenever there is a sufficient statistic u and an unbiased estimate t, not a function of u only, for a parameter p, the function $E_u t$, which is a function of u only, is an unbiased estimate for p with variance smaller than that of t. A sequential unbiased estimate for a parameter is obtained, such that when the sequential test terminates after i observations, the estimate is a function of a sufficient statistic for the parameter with respect to these observations. A special case of this estimate is that obtained by Girshick, Mosteller, and Savage (*Annals of Math. Stat.*, Vol. XVII (1946), pp. 13–23) for the parameter of a binomial distribtion.

4. **A Discussion of the Ehrenfest Model.** Preliminary report. MARK KAC, Cornell University.

A particle moves along a straight line in steps Δ, the duration of each step being τ. The probabilities that the particle at $k\Delta$ will move to the right or left are $(1/2)(1 - k/R)$

and $(1/2)(1 + k/R)$ respectively. R and k are integers and $| k | \leq R$. M. C. Wang and G. E. Uhlenbeck in their paper *On the theory of Brownian motion II* (*Rev. Mod. Phys.* Vol. 17 (1945), pp. 323–342) discuss this random walk problem and state several unsolved problems. In answer to some of the questions raised the following results are obtained: Let $(1 - z)^{R-j} \cdot (1 + z)^{R+j} = \Sigma C_k^{(j)} z^k$ (j an integer) then, the probability $P(n, m \mid s)$ that a particle starting from $n\Delta$ will come to $m\Delta$ after time $t = s\tau$ is equal to $2^{-2R}(-1)^{R+n}\Sigma(j/R)^s C_{R+j}^{(-n)} C_{R+m}^{(j)}$, where the summation is extended over all j such that $| j | \leq R$. Also, if R is even the probability $P'(n, 0 \mid s)$ that the particle starting from $n\Delta$ will come to 0 at $t = s\tau$ for the first time is calculated. For $n = 0$ this gives a solution of the so-called recurrence time problem first studied on simpler models by Smoluchowski. Through a limiting process in which $\tau \to 0$, $\Delta \to 0$, $\Delta^2/2\tau \to D$, $1/R\tau \to \beta$, $n\Delta \to x_0$, $m\Delta \to x$, $s\tau = t$, one is led to fundamental distributions concerning the velocity of a free Brownian particle. In particular, $P(n, m \mid s)$ approaches the well-known Ornstein-Uhlenbeck distribution.

5. **Sampling from Contaminated Distributions.** Preliminary report. JOHN W. TUKEY, Princeton University.

A contaminated distribution is a nearly normal distribution in which extreme observations are more frequent than in a normal distribution. By studying the bias and variability of several measures of dispersion when applied to samples from particular one-parameter families of contaminated distributions it is shown that: (i) for nearly normal distributions, the mean deviation is often better than the standard deviation; (ii) small changes in the underlying distribution may increase the sampling variance of the standard deviation by a factor of three. This suggests that, in a broad class of cases, the mean deviation is safer than the standard deviation when a single dispersion is estimated from a set of data. This conclusion need not apply in an analysis of variance situation.

6. **On the Class of Functions Defined by the Difference Equation $(x + 1)f(x + 1) = (a + bx)f(x)$.** LEO KATZ, Wayne University.

The difference equation defines only three discrete functions: the binomial, the Poisson and the Pascal functions; the first and third have one parameter (N) slightly generalized. It is shown that the Pascal function with this generalization is identical with the Polya-Eggenburgher distribution, which is a very useful form of the Compound Poisson Law and has been used to explain probability situations involving contagion. Areas for all functions in the class are given in terms of existing tables of the incomplete γ and β-functions. Observed distributions are fitted by two moments. As Carver (*Handbook of Mathematical Statistics*) pointed out, the advantages of fitting by difference equations are many; not the least is the fact that it is unnecessary to discriminate among the various functions in fitting an observed distribution. The problem of discrimination, posed by Frisch (*Metron*, Vol. 10) and others, may be resolved in terms of the sampling distribution of variances for the Poisson function, since the three functions correspond to situations where the variance is less than, equal to, or greater than the mean, respectively.

7. **Retention of Decimal Places in Matrix Calculations.** FRANKLIN E. SATTERTHWAITE, Aetna Life Insurance Company. (Read by title)

The accumulation of errors in matrix calculations has been studied by the author and others for special types of matrices and for special methods of calculation. In the present paper, error formulae are developed for the standard Doolittle and Waugh-Dwyer Compact routines. These formulae do not place any restrictions on the matrices involved and do not require any extra calculations or initial approximations. Simple rules are developed which give for each step in the calculations the number of decimal places which must be retained. These rules are efficient in the sense that the retention of fewer places will,

except for good fortune in balancing of errors, lead to results less accurate than those specified. The rules also assist in choosing that arrangement of the calculations which will lead to the smallest average number of significant figures which must be retained for the calculation as a whole.

8. **The Efficiency of the Mean Moving Range.** PAUL G. HOEL, University of California at Los Angeles. (Read by title)

The statistic $w = \sum_1^{n-1} | x_{i+1} - x_i | \sqrt{\pi}/2(n-1)$ is studied as an estimate of σ for a normal variable subject to trend effects. It is shown that the efficiency of w compares favorably with that of the mean square successive difference, δ^2. The proof that w, and also δ^2, is asymptotically normally distributed is made to depend upon a general result that can be derived from a theorem of S. Bernstein on dependent variables.

9. **Some Basic Theorems for Developing Tests of Fit for the Case of the Non-Parametic Probability Distribution Function.** BRADFORD F. KIMBALL, New York State Department of Public Service. (Read by title)

Given a universe with C.D.F. $P[X \leq x] = F(x)$. Consider a random sample of n values x_i which have been ordered so that $x_i \leq x_{i+1}$. The successive differences of the true c.d.f. values at $X = x_i$ are denoted by u_i. Thus

$$u_1 = f(x_1)$$

$$u_i = F(x_i) - F(x_{i-1}),\ 2 \leq i \leq n$$

$$u_{n+1} = 1 - F(x_n).$$

THEOREM 1. *The product power moments*

$$E(u_r^p u_s^q u_t^w \cdots)$$

for any or all different indices from 1 *to* $n+1$, *where the powers are real numbers greater than minus one, are given by*

$$E(u_r^p u_s^q u_t^w \cdots) = \frac{\Gamma(n+1)\,\Gamma(p+1)\,\Gamma(q+1)\,\Gamma(w+1)\cdots}{\Gamma(n+1+p+q+w+\cdots)}$$

COROLLARY. *If a range* $R(k, m)$ *is defined by*

$$R(0, m) = F(x_m),\ R(n+1, m) = 1 - F(x_{n+1-m})$$

$$R(k, m) = F(X_{k+m}) - F(x_k)$$

where k *and* m *are positive integers such that* $m \leq n$ *and* $k + m \leq n$, *its probability distribution is independent of* k, *and hence equal to that of* $F(x_m)$.

THEOREM 2. *Given a test funtion of* u_i

$$Y = \sum_m u_i^p$$

where p *is a real positive number, and the sum is for* m *indices chosen at random on the range* 1 *to* $n+1$. *Let* $\bar{Y}$ *and* σ^2 *denote the mean and variance of this test function. Establish a convention for increasing the indices included in the above sum for increasing* m *as* n *increases, such that* $[m/(n+1)] =$ *constant, to nearest multiple of* $1/(n+1)$. *Then the asymptotic distribution of* $(Y - \bar{Y})/\sigma$ *for inreasing* n, *subject to the above condition, is the normal distribution with zero mean and unit variance, except in the trivial case* $m = n+1$, $p = 1$.

10. **Confidence Limits for the Fraction of a Normal Population which Lies between Two Given Limits.** JACOB WOLFOWITZ, Columbia University. (Read by title)

Let $x_1, \cdots, x_N$ be N independent observations from a normal population with mean μ and variance σ^2, both unknown. Let $N\bar{x} = \Sigma x_i$ and $(N-1)s^2 = \Sigma(x_i - \bar{x})^2$ define $\bar{x}$ and s^2. Let L_1 and L_2 be given constants with $L_1 < L_2$, and let

$$\gamma = (\sqrt{2\pi}\sigma)^{-1} \int_{L_1}^{L_2} \exp -\tfrac{1}{2}\left\{\frac{y-\mu}{\sigma}\right\}^2 dy$$

By a lower confidence limit on γ with confidence coefficient α is meant a function $D(x_1, \cdots, x_N)$ such that the probability is α that $D \leq \gamma$. Since $\bar{x}$ and s^2 are sufficient estimates of μ and σ^2 the restriction that D be a function of $\bar{x}$ and s only is imposed. It is assumed that there exist a) a positive d such that $L_1 + d < \mu < L_2 - d$; b) a positive C such that $\sigma < C$. From these it follows that there exists a lower bound $G = G(d, C)$ on γ. Let $\chi^2_{1-\alpha}$ be that number for which $P\{\chi^2 < \chi^2_{1-\alpha}\} = 1 - \alpha$, where χ^2 has $N-1$ degrees of freedom, and let $w = \dfrac{\sqrt{N-1}s}{\chi_{1-\alpha}}$. It is shown that if D be defined as follows:

1) if $L_1 \leq \bar{x} \leq L_2$,

$$D = (2\pi)^{-\frac{1}{2}} \int_{L_1 - \bar{x}/w}^{L_2 - \bar{x}/w} \exp\{-\tfrac{1}{2}y^2\}\, dy$$

2) $D = G$ otherwise, then $|P\{D \leq \gamma\} - \alpha|$ approaches zero as $N \to \infty$. Thus D is a large sample lower confidence limit. The extension to upper and two-sided limits presents no difficulty.

11. **The Consolidated Doolittle Technique.** PAUL BOSCHAN, Econometric Institute. (Read by title)

The quadratic matrix notation is interpreted as a segment in a sequence of matrices wherein each successor matrix is augmented by a bordering row and column. Extension theorems based on this idea date back into the last century. The step from the original concept to one of higher order is also fruitful in discussing inverse matrices, specifically the inverse of a symmetric matrix. The symmetry of the matrix of normal equations for a set of multiple regression coefficients is restored by adding the transpose of the column on the right side of the equations, i.e. the co-variances with the dependent variable and the variance of the dependent variable itself. The inverse of this matrix can be constructed as partial sum over a series of matrices. Each individual element of this series is in itself meaningful. The solution for the set of multiple regression coefficients relating the k-th variable to the preceding $(k-1)$ variables is a column matrix. The product of this matrix with its transpose expressed in terms of the residual variance forms the k-th term in the matrix series. The summation of the first n products yields the inverse matrix. This characteristic of the inverse can be used to great advantage in the standardization of elementary computational steps.

12. **Estimation of Structural Equations through Linear Transformation of Regression Coefficients.** THEODORE W. ANDERSON and HERMAN RUBIN, Cowles Commission for Research in Economics.

A method is presented for estimating the coefficients of a single structural equation in a system $By_t' + \Gamma z_t' \equiv u_t'$ $(t = 1, 2, \cdots, T)$, where B and Γ are matrices of coefficients, y_t is a row vector of G observed jointly dependent variables, z_t of K observed predetermined

variables and u_t of G random elements. Given the distribution of the random elements, the equations define the distribution of the y_t. Some coordinates of z_t may be coordinates of y_{t-1}, etc. It is assumed that the structural equation to be estimated has at least $G - 1$ coefficients prescribed zero. The part of the population regression matrix corresponding to the predetermined variables with zero coefficients has rank one less than the number of jointly dependent variables with non-zero coefficients. The maximum likelihood estimate of this matrix is a linear transformation of the unrestricted sample regression matrix. The estimated vector of coefficients of y_t is the vector annihilated by this matrix. The vector of coefficients of z_t is estimated by means of this vector and the regression matrix. These estimates are consistent and asymptotically normally distributed. For z_t fixed, small sample confidence regions are given for the coefficients.

NEWS AND NOTICES

Readers are invited to submit to the Secretary of the Institute news items of interest

Personal Items

Dr. Armen A. Alchian, who has been discharged from the Army with the rank of Captain, is now an Assistant Professor in the Economics Dept. at the University of California at Los Angeles.

Dr. Franz L. Alt is now Assistant Director of Research at the Econometric Institute.

Colonel Dinsmore Alter is on terminal leave after more than four years' service in the Transportation Corps of the Army, and has returned to his duties as Director of the Griffith Observatory in Los Angeles. During these years Colonel Alter traveled approximately 250,000 miles on the ocean as a Transport Commander, visiting each continent except the Antarctic.

Dr. Theodore W. Anderson, formerly with the Cowles Commission, is now an Instructor in the Dept. of Math. Statistics at Columbia University, and plans to be on a Guggenheim Fellowship beginning in June 1947.

Mr. Herbert Barkan has been appointed to an Instructorship in the Newark College of Engineering.

Mr. Robert E. Bechhofer, formerly a statistician with The Kellex Corporation, is a graduate student at Columbia University this year.

Mr. Stanley G. Behrends is now Cost Accountant with the California Wire Cloth Corporation, in Oakland.

Messrs. Carl A. Bennett, Jack I. Northam, and Max A. Woodbury have all returned from various types of war service to the University of Michigan as graduate students in statistics. Mr. Bennett was with the Manhattan Engineering District for over two years, first at the Metallurgical Lab., University of Chicago, and then at Oak Ridge, Tenn. Mr. Northam was recently discharged from the Army with the rank of Lieutenant, having served with the Signal Corps for four years in the Pacific area. Mr. Woodbury was discharged from the Army with the rank of Captain, having been in the Meteorology service for five years, most of which time was spent in the European theater.

Mr. Richard Berger has received his discharge from the Navy, and is employed as a Research Analyst with Dun and Bradstreet.

Dr. Archie Blake, formerly at Aberdeen Proving Ground, is now Senior Statistician in the Office of the Army Surgeon General.

Dr. Ernest E. Blanche, who had been teaching in one of the European Army University Centers, is now Principal Administrative Analyst in the Plans and Policy Office of the War Department General Staff, and is also Lecturer at American University.

Mr. Royal F. Bloom has resigned the position which he held for a short time with the Psychology Dept. of Iowa State College after his release from the

Navy last March, and has returned to the Navy Department as Assistant Head of the Classification Research Division, Bureau of Naval Personnel.

Mr. Earl K. Bowen has been appointed to an Instructorship in statistics at Babson Institute of Business Administration.

Mr. Albert H. Bowker is enrolled this year as a graduate student at the University of North Carolina.

Mr. Charles R. Brearty has joined the Technical Staff of Bell Telephone Laboratories, Inc.

Mr. Clyde A. Bridger is on leave from his position at the University of Utah, and is spending the year at the Institute of Statistics in Raleigh, North Carolina.

Mr. Arthur W. Brown, formerly with the Columbia University Division of War Research, is now with the Standard Oil Company of New Jersey.

Dr. George W. Brown, formerly connected with the RCA Laboratories at Princeton as Research Engineer, has accepted a position as Research Associate Professor in the Statistical Laboratory at Iowa State College.

Mr. Richard H. Brown has been appointed to a Lectureship in Mathematics at Columbia University.

Dr. Richard S. Burington, Director of the Evaluation and Analysis Groups of the Research and Development Division of the Bureau of Ordnance, Navy Department, has been named Chief Mathematician, Bureau of Ordnance.

Mr. Roy A. Chapman, who has been Silviculturist at the Hitchiti Experimental Forest, Round Oak, Georgia, is now with the U. S. Forest Service in Washington, D. C.

Dr. Way Ming Chen has been appointed to an Instructorship in mathematics at Brown University.

Dr. John M. Clarkson has been promoted to a professorship at North Carolina State College.

Mr. S. Lee Crump has been promoted to an Assistant Professorship at Iowa State College.

Dr. Joseph F. Daly, formerly an Instructor at Catholic University, and more recently a Lieutenant in the Navy Department, is now Statistician with the Bureau of the Census.

Dr. Daniel B. DeLury has been promoted to a professorship in statistics at Virginia Polytechnic Institute.

Dr. Acheson J. Duncan has been appointed to an associate professorship of political economy at The Johns Hopkins University.

Dr. Jack W. Dunlap, formerly at Rochester University and more recently a Lieutenant Commander in the U. S. Navy, is now Director of the Division of Biomechanics of the Psychological Corporation.

Mr. Francis B. Elmore has been discharged from the Army and is Quality Control Engineer at the Union Bag and Paper Company, in Savannah, Ga.

Mr. Mark W. Eudey has returned from service to his former position with the Statistical Laboratory at the University of California.

Mr. Charles D. Ferris, formerly at Aberdeen Proving Ground, is now Quality Control Engineer with the General Electric Company, in Bridgeport, Conn.

Mr. Lester R. Frankel is now a statistician with Dun and Bradstreet.

Mr. John E. Freund has accepted a position as assistant professor of mathematics at Alfred University.

Dr. Bernard Friedman has been promoted to an assistant professorship at New York University.

Dr. Milton Friedman has been appointed to an associate professorship in the Department of Economics at the University of Chicago.

Mr. G. Rupert Gause is now with the Technical Staff of the Bell Telephone Laboratories.

Professor Edwin L. Godfrey has been appointed Head of the Department of Mathematics and Astronomy at Defiance College.

Dr. Casper Goffman has been appointed to an assistant professorship in the Department of Mathematics at the University of Kentucky.

Mr. Harry H. Goode is now a Mathematician in the Office of Research and Inventions, U. S. Navy.

Mr. Robert D. Gordon is a Teaching Assistant in Mathematics at Indiana University.

Mr. Bert A. Gottfried has returned from the service and is Research Analyst with Dun and Bradstreet.

Dr. Clyde H. Graves, formerly at Pennsylvania State College, is now Operations Branch Chief of the Office of Price Board Management, OPA.

Dr. Joseph A. Greenwood has recently been separated from active duty with the Navy and is now a statistician in the Bureau of Aeronautics.

Mr. Harris T. Guard has returned to Colorado A. and M. as an Instructor in the Department of Mathematics.

Dr. Joy P. Guilford has returned to his former position as Professor of Psychology at the University of Southern California.

Prof. Emil J. Gumbel, formerly with the New School of Social Research, has been appointed to a Special Lectureship in Statistics at Newark College of Engineering.

Mr. Lee S. Gunlogson has been discharged from the Navy and is now in the statistical department of the Lumbermens Mutual Casualty Company, Chicago.

Dr. Paul R. Halmos has been appointed to an assistant professorship in mathematics at the University of Chicago.

Professor Preston C. Hammer has returned to his former position at Oregon State College.

Mr. Joseph O. Harrison, Jr. is now employed as a mathematician for the Harvard University Automatic Sequence Controlled Calculator Project in Cruft Laboratory.

Mr. Millard Hastay, formerly with the Statistical Research Group at Columbia University, is now Research Associate at the National Bureau of Economic Research.

Mr. Bernard Hecht has been promoted from Chief Quality Control Engineer to Manager of the Quality Control Department of the International Resistance Company, Philadelphia.

Mr. Joseph L. Hodges, Jr. has been appointed to a teaching assistantship in mathematics at the University of California.

Dr. Paul G. Hoel has been promoted to an associate professorship in mathematics at the University of California at Los Angeles.

Mr. Richard A. Hornseth has been appointed to an instructorship in the Department of Sociology and Anthropology at the University of Wisconsin.

Mr. Harry M. Hughes has been appointed to a teaching assistantship at the University of California.

Mr. Leonid Hurwicz, formerly with the Cowles Commission, has been appointed to an associate professorship at Iowa State College.

Mr. Joseph B. Jeming has been separated from service with the Air Forces, and is now a Financial and Economic Consultant in New York City.

Mr. Paul Johner has been discharged from the Army and is now in the Industrial Engineering Division of the Aluminum Company of America, New Kensington, Pa.

Miss Margaret Kampschaefer, who is a statistician in the War Department, is now serving in the Supply Division of the Air Force Service Command in Erlangen, Germany.

Dr. Leo Katz has been appointed to an assistant professorship at Michigan State College.

Mr. Frederick G. King has been discharged from the Army and is now a civilian instructor in the Anti-Aircraft Artillery School at Fort Bliss.

Dr. Tjalling Koopmans has been appointed Associate Professor of Economics at the University of Chicago.

Mr. Paul J. Kopp has been discharged from the Army and is now with the Patent Department of the Gulf Oil Corporation, Washington, D. C.

Dr. Carl F. Kossack has accepted a position as mathematician with the Joint Army-Navy Air Intelligence in the Strategic Vulnerability Branch.

Dr. Waclaw Kozakiewicz has been promoted to an assistant professorship in mathematics at the University of Saskatchewan.

Professor Rafael Laguardia has returned to Uruguay as Director of the Institute de Matematica y Estadistica, Facultad de Ingenieria.

Dr. Charles R. Langmuir, formerly with the Psychological Corporation, is now Secretary-Treasurer and Lab. Director of the Bennett and Langmuir Development Corporation, Mamaroneck, N. Y.

Mr. Charles M. Larson has accepted a position as mathematician with the Pacific Mutual Life Insurance Company, Los Angeles.

Miss Lucy A. LaSala, formerly with the research group at Columbia University, is now teacher of mathematics at East New York Vocational High School.

Dr. Richard A. Leibler is now a Member of the Institute for Advanced Study, Princeton.

Miss Grace L. Lesser, formerly with the research group at Columbia University, is now employed as a statistician with the Econometric Institute.

Miss Myra Levine has accepted a position as statistician with the Socony-Vacuum Oil Company, in New York City.

Dr. Jerome C. R. Li has been appointed to an instructorship at Oregon State College.

Professor William T. Martin has accepted a professorship in the Department of Mathematics at Massachusetts Institute of Technology.

Miss Ethelyne L. McBee, formerly with the U. S. Department of Agriculture, is now teaching science and mathematics at the Falls Church High School, Falls Church, Virginia.

Dr. Paul W. McGann has been appointed to an assistant professorship in economics at American University.

Dr. Max F. Millikan has been appointed to a research associateship at Yale University.

Mr. Probodh C. Mittra has accepted a position as consulting statistician with the United Nations Economic and Social Council.

Dr. Marjorie E. Moore has transferred from her position as statistician with the Social Security Administration, to one as Program Analyst in the Office of Vocational Rehabilitation, Federal Security Agency.

Miss Judith Moss, who was with the research group at Columbia University, is now research assistant with the National Bureau of Economic Research.

Dr. Frederick Mosteller has been appointed to a lectureship and research associateship in the Department of Social Relations at Harvard University.

Mr. James E. Myers, formerly with the Naval Research Laboratory at Anacostia Station, is now with the research group of the Moore School of Electrical Engineering, University of Pennsylvania.

Mr. Stanley W. Nash is a graduate student this year at the University of California.

Professor J. Neyman is on leave from his position at the University of California for the fall semester, and is Visiting Professor of Mathematical Statistics at Columbia University.

Mr. Russell T. Nichols has been discharged from the Army, and is a graduate student at the University of Chicago.

Mr. Harold Nisselson has been discharged from the Navy, and is now a statistician in the Bureau of the Census, where he was formerly employed.

Professor Nilan Norris has been separated from his service with the Army, with the rank of Major, and has returned to his position in the Department of Economics at Hunter College.

Dr. Guy H. Orcutt, formerly at Massachusetts Institute of Technology, has accepted a research position in the Department of Applied Economics, Cambridge University. This new department is to be modelled somewhat along the lines of the Cowles Commission at the University of Chicago, and is to be under the direction of Dr. J. R. N. Stone.

Mr. Warren H. Page has been separated from service with the Army, and is now a graduate student at Columbia University.

Mr. Nicholas Pastore has been appointed to an instructorship at Union Junior College, Cranford, New Jersey.

Mr. I. B. Perrott has been demobilized from the British Army with the rank of Major.

Mr. George W. Petrie, III has accepted a position as Special Engineer with the Bethelehem Steel Company.

Dr. Harry S. Pollard has been promoted to a professorship at Miami University.

Dr. G. Baley Price, Professor of Mathematics at the University of Kansas, has been awarded a Post-Service Guggenheim Fellowship, beginning September 1, 1946.

Mr. Robert J. Randall has been discharged from the Army and is now a graduate student at Columbia University.

Professor Lowell J. Reed, of the School of Hygiene and Public Health, The Johns Hopkins University, has been appointed Vice-President of the University.

Dr. Francis Regan has been promoted to a professorship at St. Louis University.

Mrs. Kathryn B. Rolfe, formerly at the University of California at Berkeley, has accepted a position as associate in mathematics at the University of California College of Agriculture, at Davis.

Mr. Frank Saidel is a graduate student in mathematical statistics this year at Columbia University.

Dr. Leonard J. Savage has been awarded a Special Rockefeller Fellowship, beginning September 1946.

Professor Henry Scheffé of the University of California at Los Angeles has been awarded a Guggenheim Fellowship, and is spending the year at the University of California at Berkeley.

Professor Andrew S. Schultz, Jr. has been separated from service with the Army and has returned to Cornell University with the rank of associate professor.

Dr. Saul B. Sells, formerly with the OPA, has accepted a position as Assistant to the President of the A. B. Frank Company, San Antonio.

Mr. Lawrence W. Shaw is now a statistician with the U. S. Public Health Service in Bethesda.

Dr. Ronald W. Shephard has been appointed to a lectureship at the University of California, Berkeley.

Mr. Clifford R. Simms has accepted a position as manager of the Cleveland office of the B. E. Wyatt Company.

Mr. George B. Simon has been separated from Army service with the rank of major, and has accepted a civilian position as chief of the Analysis and Research Unit, Psychological Section, Office of Surgeon, Barksdale Field.

Mr. Herbert Solomon has been appointed to an instructorship at the College of the City of New York.

Mr. Melvin D. Springer has returned to the University of Illinois and has been appointed to an assistantship.

Mr. Andrew P. Stergion has been discharged from the Army, and is now Statistical and Quality Control Engineer with the Corning Glass Works.

Mr. Milton S. Stevens has been discharged from the Navy, and has accepted a position as Director of Special Projects with Time, Inc.

Dr. George J. Stigler has been appointed to a professorship in economics at Brown University.

Mr. Alexander L. Stott has been discharged from the Navy, and is now a staff assistant in the Treasury Department of the American Telephone and Telegraph Company.

Dr. L. V. Toralballa has accepted a teaching position at Fordham University.

Dr. Walter R. Van Voorhis has returned to Fenn College, with the rank of associate professor.

Mr. Edward H. Van Winkle has been appointed to a professorship of business statistics at Rensselaer Polytechnic Institute.

Dr. Charles W. Vickery has been appointed to an associate professorship at Ohio State University.

Mr. David F. Votaw, Jr. has been separated from service with the Navy and has returned to Princeton University as Research Associate.

Mr. W. Allen Wallis has been appointed to a professorship at the University of Chicago.

Mr. Ralph E. Wareham is now managing director of the National Photocolor Corporation.

Dr. Jacob Wolfowitz has been appointed to an associate professorship in mathematical statistics at Columbia University.

Mr. John F. Wyckoff, formerly at Trinity College, has accepted a position in the Research Division of the Actuarial Department, Connecticut General Life Insurance Company, Hartford.

Mr. Earl K. Yost, Jr. has been appointed to a graduate assistantship in mathematics at the University of Oregon.

A conference on applied mathematical statistics was held at Lake Junaluska, North Carolina, August 4–9, 1946 under the sponsorship of the Institute of Statistics of the University of North Carolina. The following individuals attended the conference: C. I. Bliss, W. G. Cochran, Gertrude M. Cox, D. B. Duncan, C. Eisenhart, R. A. Fisher, Carl F. Kossack, Frederick Mosteller, H. W. Norton, Paul Peach, Charles F. Roos, Walter A. Shewhart, Frederick Stephan, Gerhard Tintner, John W. Tukey, S. S. Wilks, C. P. Winsor, and J. Wolfowitz.

Newark College of Engineering is sponsoring a series of conferences on Industrial Statistics. The first of these, on Acceptance Sampling, began on September 27 and ran for eleven four-hour Friday sessions. Among the members of the Advisory Panel on Industrial Statistics are Institute members S. B. Littauer, A. I. Peterson, W. A. Shewhart, and S. S. Wilks.

New Members

The following persons have been elected to membership in the Institute:

Anscombe, F. J. Rothamsted Experimental Station, Harpenden, Herts, Eng.

Back, Kurt W., M.A. (California at L. A.) Stat., Surveillance Branch, Ballistic Res. Lab., Aberdeen Proving Gd., Md.

Bresnahan, Maurice F. Stat., U. S. Bur. of Labor Statistics, Wash., D. C., *Apt. 305, 1015 N St., N.W., Wash. 1*

Chung, Kai-Lai, M.A. (Princeton) Graduate Coll., Princeton Univ., Princeton, N. J.

Clarke, P. C. Asst. Gen. Mgr., Hunter Pressed Steel Co., Lansdale, Pa., *Line Lexington, Pa.*

Coon, Helen J., M.A. (Southern Methodist) Ballistic Res. Lab., Aberdeen Proving Gd., Md.

Copp, Warren F., B.S. (Ohio State) Supv., Quality Control Dept., Wheeling Steel Corp., Yorkville Works, Yorkville, Ohio

Divatia, Vasishtha V., B.Sc. (Bombay) Student in Math. Stat., Columbia Univ. *#724 John Jay Hall, Columbia Univ., N. Y. City.*

Fanshaw, Hugh L., M.S. (Manitoba) Standards Supv., Canadian Indus. Ltd., General Chemicals Div., Hamilton, Ont., Can., *120 St. Clair Ave.*

Feriet, Kampe de, Dr. Sci. (Paris) Professeur a la Faculte des Sci. de l'Universite de Lille, 16 rue des Jardins, Lille, France

Fine, Clarence B., B.S.S. (C.C.N.Y.) Economist, OPA, Wash., D. C., *1388 Tuckerman St., N.W., Wash. 11*

Golub, Abraham, B.A. (Brooklyn) Math., Ballistic Res. Lab., Aberdeen Proving Gd., Md., *Men's Dorm*

Gomberg, William, Ph.D. (Columbia) Dir. of Mgt., Engr. Dept., International Ladies Garment Workers Union, 1710 Broadway, N. Y., N. Y., *444 Beach 142nd St., Neponsit, L. I.*

Halton, Frederick J., Jr. Asst. to Pres., John Deere & Co., 230 S. Clark St., Chicago, Ill., *1314 Westview Rd., Highland Park*

Hanson, Robert H., M.S. (Iowa) Stat., Bur. of the Census, Wash., D. C., *3148 Westover Dr., Wash. 20*

Hardy, Philip H., B.S. (Rice) Quality Engr., General Elec. Co.; on leave, Cpl. US Army, 4000 BU Sq. S., Wright Field, Dayton, Ohio

Hasty, Willis L., Jr., B.C.S. (Benjamin Franklin) Capt., Signal Corps, *3475 South Wakefield St., Arlington, Va.*

Hess, Ida I., A.B. (Indiana) Stat., Population Div., Bur. of the Census, Wash., D. C., *1425 Rhode Island Ave., N.W.*

Jacobson, Jack J., M.B.A. (Chicago) Stat., Spiegel, Inc., Chicago, Ill., *3562 W. Palmer St.*

Janko, Prof. Jaroslav, Technical Univ., Prague, Czechoslovakia, *Na bojisti 3, Praha II*

Karp, Abraham E., M.S. (C.C.N.Y.) Stat., Aberdeen Proving Gd., Md., *25 Aberdeen Ave.*

Keefe, David P., B.S. (St. Thomas) Supv., Raw Material Testing, Minn. Mining and Mfg. Co., St. Paul 6, Minn., *590 Holly Ave., St. Paul*

Kellogg, Lester S., M.A. (Northwestern) Chief, Prices and Cost of Living Branch, Bur. of Labor Statistics, U. S. Dept. of Labor, Wash., D. C., *404 Shady Lane, Falls Church, Va.*

Kindig, Fred E., B.S. (Pennsylvania State) Industrial Math., Westinghouse Elec. Co., Braddock Ave., E. Pittsburgh, Pa., *53 Nantucket Dr., R.D. 6, Pleasant Hills, Pittsburgh 10*

Kosambi, D. D., ScB. (Harvard) Dept. of Math., Tata Inst. of Fundamental Res., 53 Pedder Rd., Bombay, India

Kruskal, William, M.S. (Harvard) *150 W. 30th St., N. Y., N. Y.*

Lacey, Prof. Oliver L., Ph.D. (Cornell) Head, Dept. of Psychology, Univ. of Ala., University, Ala.

Martin, Cyrus A., B.S. (Iowa State) Stat., Headquarters Fifth Army, *7817 South Shore Dr., Chicago 49, Ill.*

Maxwell, Pat, Jr., B.S. (Fordham) Stat., Surveillance Br., Ballistic Res. Lab., Aberdeen Proving Gd., Md., *Apt. A6-3, Baldwin Manor*

Mills, Vicente, P.O. Box 2090, Manila, Philippine Islands

Newman, Doris, M.A. (Michigan) Instr. in Stat., School of Bus. Adm., Crosby Hall, Univ. of Buffalo, Buffalo 14, N. Y.

Ott, Asso. Prof. Ellis R., Ph.D. (Illinois) Dept. of Math., Rutgers Univ., New Brunswick, N. J., *199 Sterling Dr., Orange*

Paull, Allan E., B.A. (Manitoba) 114 Mangum, U. of N. C., Chapel Hill, N. C.

Pollard, A. H. The Mutual Life & Citizens' Ass. Co. Ltd., Martin Place and Castlereath St., Sydney, N.S.W., Australia

Quenouille, Maurice H., B.A. (Cambridge) Stat., Rothamsted Experimental Station, Harpenden, Herts, Eng., *8 Chipstead St., Fulham, London, S.W. 6*

Ryan, Asso. Prof. Thomas A., Ph.D. (Cornell) Dept. of Psychology, Morrill Hall, Cornell Univ., Ithaca, N. Y

Sachs, Rose, B.A. (Goucher) Stat., *1428 R St., N.W., Wash. 9, D. C.*

Sandelius, D. Martin, Fil.kand. (Stockholm) Baltzar von Platensg. 5, VI, Stockholm, Sweden

Schoenbaum, Prof. Emil, Ph.D. (Prague) Dept. of Applied Math., School of Natural Science, Univ. of Charles IV, Prague, Czechoslovakia, *Praha XIX, Terronska 26*

Schweitzer, Morton D., Ph.D. (Columbia) Stat. and econ. consultant, 60 West 10th St., N. Y. 11, N. Y.

Shybekay, Derso S., Ph.D. (Budapest) Pres., Indus. Res. Council, P. O. Box 324, Minneapolis 1, Minn.

Stein, Irving, B.S. (Mass. Inst. Tech.) Qual. Control Engr., Polaroid, Inc., *10 Forest Park Ave., Adams, Mass.*

Stephens, William H., M.Sc. (Queen's) British Commonwealth Scientific Office, *1785 Massachusetts Ave., Wash., D. C., 3807 Military Rd., N.W.*

Strieby, J. Glenn, B.S. (Iowa Wesleyan) Tech. Dept., Kimberly-Clark Corp., Neenah, Wis., *616 E. Circle St., Appleton*

Thompson, Juanita, Qual. Control Inspector, *4117 West 25th St., Chicago 23, Ill.*

Thompson, Sidney L., M.S. (Tulane) Ala. Polytechnic Inst., Auburn, Ala.

Tompkins, Haldan M. Head, Works Control Lab., National Carbon Co., Fostoria, Ohio, *P.O. Drawer 191*

Tosti, Carlo R., M.A. (Stanford) Capt., AC, Adm. Engr., Tech. Control Officer, Wright Field, Dayton, Ohio, *501 Forest Ave., Dayton 5*

Toulouse, Julian H., Ph.D. (Iowa State) Chief Engr., Quality and Specifications Sec., Owens-Ill. Glass Co., Toledo, Ohio, *P. O. Box 1035–1036, Toledo 1*

Truksa, Ladislav, Ph.D. (Charles Univ.) Dir., Lecturer in Math. Stat., Charles Univ., Praha XI-1800, Czechoslovakia

Urie, Frank D., A.B. (Michigan) Supt. of Inspection, Elgin National Watch Co., Elgin, Ill.

Vajda, Stefan, Ph.D. (Vienna) Stat., Admiralty, Whitehall, London, Eng., *54 Chapel Way, Epsom, Surrey*

Weiss, Lionel, M.A. (Columbia) Gottsberger Fellow, Columbia Univ., N. Y., N. Y., *2205 Davidson Ave., N. Y. 53*

Weyl, Eric, (Cologne) Textile Consultant, Chicopee Mfg. Corp., Manchester, N. H., *279 Orange St.*

REPORT ON THE ITHACA MEETING OF THE INSTITUTE

The Ninth Summer Meeting of the Institute of Mathematical Statistics was held at Cornell University, Ithaca, New York, on Thursday, August 21, and Saturday, August 23, 1946. The meeting was held in conjunction with the summer meetings of the American Mathematical Society and the Mathematical Association of America. The following 71 members of the Institute attended the meeting:

P. L. Alger, C. B. Allendoerfer, T. W. Anderson, Jr., J. L. Barnes, E. E. Blanche, Paul Boschan, A. H. Bowker, A. E. Brandt, R. S. Burington, W. G. Cochran, E. P. Coleman, H. B. Curry, J. H. Curtiss, J. L. Doob, J. Dutka, P. S. Dwyer, B. Epstein, Will Feller, C. D. Ferris, R. M. Foster, J. E. Freund, M. A. Girschick, A. A. Goodman, Louis Guttman, W. W. Gutzman, P. R. Halmos, T. E. Harris, Bertha I. Hart, E. H. C. Hildebrandt, P. G. Hoel, R. H. Hoskins, Harold Hotelling, W. W. Jacobs, T. J. Jaramillo, Evan Johnson, Jr., H. L. Jones, Mark Kac, Irving Kaplansky, Leo Kart, Tjalling Koopmans, C. F. Kossack, M. M. Lavin, Walter Leighton, Jr., Howard Levene, M. S. Macphail, J. W. Mauchly, P. J. McCarthy, E. C. Molina, Margaret E. Moore, J. E. Morton, L. F. Nanni, P. M. Neurath, E. G. Olds, G. B. Price, C. J. Rees, Selby Robinson, Herman Rubin, P. J. Rulon, Arthur Sard, F. E. Satterthwaite, I. E. Segal, G. R. Seth, Andrew Sobczyk, Herbert Solomon, C. M. Stein, F. F. Stephan, A. P. Stergion, A. W. Tucker, J. W. Tukey, J. L. Ullman, Abraham Wald, S. S. Wilks.

The first session, a joint session with the American Mathematical Society, was held on Thursday morning, and was devoted to contributed papers. Professor W. G. Cochran, President of the Institute, presided. The following seven papers were presented:

1. *A Test of Randomness in Two Dimensions.*
   Mr. Howard Levene, Columbia University.
2. *Asymptotic Distribution of Moments from a System of Linear Stochastic Difference Equations.*
   Mr. Herman Rubin, Cowles Commission for Research in Economics.
3. *Conditional Expectation and Unbiased Sequential Estimation.*
   Professor David Blackwell, Howard University.
4. *A Discussion of the Ehrenfest Model.* Preliminary report.
   Professor Mark Kac, Cornell University.
5. *Sampling from Contaminated Distributions.* Preliminary report.
   Professor John W. Tukey, Princeton University.
6. *On the Class of Functions Defined by the Difference Equation* $(x + 1)\, f(x + 1) = (a + bx)\, f(x)$.
   Dr. Leo Katz, Wayne University.
7. *Retention of Decimal Places in Matrix Calculations.*
   Dr. Franklin E. Satterthwaite, Aetna Life Insurance Company.

The following four papers were presented by title:

8. *The Efficiency of the Mean Moving Range.*
   Professor Paul G. Hoel, University of California at Los Angeles.
9. *Some Basic Theorems for Developing Tests of Fit for the Case of the Non-Parametric Probability Distribution Function.*
   Mr. Bradford F. Kimball, N. Y. State Department of Public Service, New York City.

10. *Confidence Limits for the Fraction of a Normal Population Which Lies Between Two Given Limits.*
    Professor Jacob Wolfowitz, Columbia University.
11. *The Consolidated Doolittle Technique.*
    Dr. Paul Boschan, The Econometric Institute, Inc.

Abstracts of all these papers appear elsewhere in this issue of the *Annals.*

At two o'clock on Thursday afternoon there was a joint session with the American Mathematical Society which featured the invited address of Professor J. L. Doob of the University of Illinois on *Probability in Function Space.* This address was followed by a business meeting of the Institute which featured reports by the President, the Secretary-Treasurer, the Editor, and Professor Feller, who spoke for the recently created committee on the distribution of the *Annals* in the war areas.

On Thursday evening there was a joint dinner with the American Mathematical Society and the Mathematical Association of America.

The meeting closed with a session on Friday morning devoted to the topic, *Multivariate Analysis for Non-Experimental Data.* Professor Will Feller, of Cornell University, presided. Professor T. Koopmans, of the Cowles Commission for Research in Economics, presented a paper entitled *Statistical Inference in Dynamic Economic Models.* Dr. T. W. Anderson, Jr. presented a paper written by himself and Mr. Herman Rubin entitled *Estimation of Structural Equations through Linear Transformation of Regression Coefficients.* The meeting concluded with a discussion of these papers.

P. S. DWYER,
Secretary.

REPORT OF THE PRINCETON MEETING OF THE INSTITUTE

The twenty-third meeting of the Institute of Mathematical Statistics was held in Princeton, New Jersey on Friday, November 1, 1946, in connection with the year-long Celebration of the Bicentennial of Princeton University. The meeting was devoted entirely to *Analysis of Variance*. The meeting was attended by 118 persons including the following 96 members of the Institute:

Adam Abruzzi, Forman S. Acton, R. L. Anderson, T. W. Anderson, Jr., M. S. Bartlett, Robert Bechofer, Gilbert W. Beebe, J. H. Bigelow, Archie Blake, C. I. Bliss, A. E. Brandt, Burton H. Camp, George C. Campbell, A. George Carlton, Kai Lai Chung, W. G. Cochran, Gertrude Cox, Harold Cramér, S. Lee Crump, J. H. Curtiss, Joseph F. Daly, Besse B. Day, D. B. DeLury, V. V. Divatia, J. Dutka, Churchill Eisenhart, B. Epstein, H. L. Fanshaw, Nicholas Fattu, Will Feller, Merrill M. Flood, Bernard Friedman, Hilda Geiringer, H. H. Goldstine, Joseph A. Greenwood, E. J. Gumbel, Margaret Gurney, L. Gutmann, T. E. Harris, Millard Hastay, Irwin S. Hoffer, C. J. Kirchen, B. F. Kimball, Lila F. Knudsen, H. S. Konijn, Jack Laderman, J. D. Maddrill, Sophie Marcuse, H. C. Mathisen, J. W. Mauchly, Margaret Merrell, Elmer B. Mode, Margaret E. Moore, J. E. Morton, Judith Moss, F. Mosteller, Charles M. Mottley, Ray B. Murphy, P. M. Neurath, Hugo Nilson, Gottfried E. Noether, Monroe L. Norden, H. W. Norton, C. O. Oakley, P. S. Olmstead, J. G. Osborne, Ellis R. Ott, C. J. Rees, W. A. Reynolds, A. C. Rosander, David Rosenblatt, Ernest Rubin, P. U. Rulon, Frank Saidel, Marian M. Sandomire, Walter A. Shewhart, James G. Smith, Milton Sobel, Herbert Solomon, Mortiner Spiegelmon, Charles M. Stein, G. R. Stibitz, John R. Tomlinson, Marion M. Torrey, John W. Tukey, D. F. Votaw, Jr., F. M. Wadley, Alton J. Wadman, A. Wald, Robert M. Walter, Lionel Weiss, Frank Wilcoxon, S. S. Wilks, C. P. Winsor, J. Wolfowitz, and W. J. Youden.

At the morning session the following program was presented with Professor S. S. Wilks of Princeton University as chairman:

Topic: *Mathematical Approaches to the Analysis of Variance*

Papers: *Two Probability Models for the Analysis of Variance*
Professor A. Wald, Columbia University
Applications of Analysis of Variance
Professor M. S. Bartlett, Cambridge University and The University of North Carolina

Discussion: Professor S. L. Crump, Iowa State College
Dr. J. F. Daly, Bureau of the Census
Professor J. W. Tukey, Princeton University
Professor C. P. Winsor, Johns Hopkins University
Professor J. Wolfowitz, Columbia University

The program for the afternoon session, under the chairmanship of Professor Will Feller of Cornell University, was as follows:

Topic: *Multivariate Problems in the Analysis of Variance*

Papers: *Analysis of Covariance*
Professor W. G. Cochran, The University of North Carolina
Vector Methods
Professor J. W. Tukey, Princeton University

Discussion: Professor T. W. Anderson, Columbia University
Professor C. I. Bliss, Yale University
Professor Harold Cramér, The University of Stockholm and Princeton University
Professor D. B. DeLury, Virginia Polytechnic Institute
Professor P. L. Hsu, The University of North Carolina

The evening session consisted of round table discussion on *Unsolved Problems of the Analysis of Variance*, with Professor Gertrude M. Cox as chairman.

Members of the Institute and others who attended the meeting were guests of the Institute for Advanced Study at tea in Fuld Hall from 4 to 6 P.M. Those attending the evening session were guests of Princeton members of the Institute for refreshments in Fine Hall from 10 to 11 P.M.

P. S. DWYER,
Secretary.

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(As of October 1, 1946)

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